Chain Rule

Single variable function

$$\frac{d}{dx} f(g(x)) = \left.\frac{df}{dy}\right|_{y=g(x)} \cdot \frac{dg(x)}{dx}$$

Two variables function $z = f(x, y)$

$$\frac{d}{dt} f(x(t), y(t)) = \lim_{t \to t_0} \frac{\Delta z}{\Delta t}$$

$t - t_0$

$$\left(\Delta z = f(x(t), y(t)) - f(x(t_0), y(t_0)) \right)$$

(assume f is differentiable)

$$= \lim_{\substack{\Delta t \to 0 \\ (t \to t_0)}} \left(f_x(x(t), y(t)) + \varepsilon_1\right) \frac{\Delta x}{\Delta t} + \left(f_y(x(t), y(t)) + \varepsilon_2\right) \frac{\Delta y}{\Delta t}$$

$(*)$

When $\Delta t \to 0$ $(t \to t_0)$

we have $(x(t), y(t)) \Rightarrow (x(t_0), y(t_0))$

(assume $x(t)$, $y(t)$ are also differentiable at t_0, therefore continuous at t_0)

$\Rightarrow (\Delta x, \Delta y) \longrightarrow (0, 0)$

$$\Rightarrow \lim_{\Delta t \to 0} \varepsilon_1 = 0 = \lim_{\Delta t \to 0} \varepsilon_2$$

$\therefore$ (*)

$$= f_x(x(t_0), y(t_0)) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$+ f_y(x(t_0), y(t_0)) \lim_{\Delta t} \frac{\Delta y}{\Delta t}$$

$$= f_x(x(t_0), y(t_0))\, x'(t_0) + f_y(x(t_0), y(t_0))\, y'(t_0)$$

In other words

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$$

Similarly, $\partial_s f(x(s,t), y(s,t), z(s,t))$

$$= f_x(x(s,t), y(s,t), z(s,t)) \cdot \partial_s x(s,t)$$
$$+ f_y(\qquad) \cdot \partial_s y(s,t)$$
$$+ f_z(\qquad) \, \partial_s z(s,t)$$

$\partial_t f(x(s,t), y(s,t), z(s,t))$

$$= f_x(\; \cdots \;) \cdot \partial_t x(s,t)$$
$$+ f_y(\; \cdots \;) \, \partial_t y(s,t)$$
$$+ f_z(\qquad) \, \partial_t z(s,t)$$

Implicit differentiation revisited

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $F(x,y,z)=0$

(i.e. If $z(x,y)$ is implicitly defined by $F(x,y,z)=0$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$)

<u>Sol</u> $F(x,y,z(x,y))=0$ (as two functions of (x,y))

$$\Rightarrow \partial_x F(x,y,z(x,y))=0$$

$$\Rightarrow \partial_1 F\cdot\underbrace{\frac{\partial x}{\partial x}}_{=1}+\partial_2 F\cdot\underbrace{\frac{\partial y}{\partial x}}_{=0}+\partial_3 F\cdot\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \partial_1 F(x,y,z(x,y))+\partial_3 F(x,y,z(x,y))\cdot\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \frac{\partial z}{\partial x}\text{ at }(x,y,z)=-\frac{\partial_1 F(x,y,z)}{\partial_3 F(x,y,z)}=-\frac{\partial_x F(x,y,z)}{\partial_z F(x,y,z)}$$

Similarly

$$\frac{\partial z}{\partial y} \text{ at } (x,y,z) = -\frac{\partial_y F(x,y,z)}{\partial_z F(x,y,z)}$$

Eg 1 Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(1,1,1)$

from $F(x,y,z) = xy + z^3x - 2yz = 0$

Ans. $\partial_x F(1,1,1) = y + z^3 \Big|_{(1,1,1)} = 2$

$\partial_y F(1,1,1) = x - 2z \Big|_{(1,1,1)} = -1$

$\partial_z F(1,1,1) = 3z^2x - 2y \Big|_{(1,1,1)} = 1$

$\therefore \frac{\partial z}{\partial x}$ at $(1,1,1) = -\frac{2}{1} = -2$

$\frac{\partial z}{\partial y}$ at $(1,1,1) = -\frac{(-1)}{1} = 1$

Directional derivative

Def: $\left(\frac{df}{ds}\right)_{\vec{u}, P_0}$ $\left(= D_{\vec{u}} f(x_0, y_0)\right)$
$(P_0 = (x_0, y_0))$

(Derivative of $f(x, y)$ at $P_0 = (x_0, y_0)$ in the direction of the <u>unit vector</u> $\vec{u} = (u_1, u_2)$
$(u_1^2 + u_2^2 = 1)$)

$$\stackrel{\text{def}}{=\!=} \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s - 0}$$

$$\left(= \lim_{s \to 0} \frac{f(x(s), y(s)) - f(x(0), y(0))}{s - 0} \right.$$

where $x(s) = x_0 + su_1$, $y(s) = y_0 + su_2$)

Thm If $f(x,y)$ is differentiable at (x_0,y_0), then $D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\vec{u}$

where $\nabla f = (f_x, f_y)$ (gradient of f)

pf: $\Delta z = (f_x(x_0,y_0)+\varepsilon_1)\Delta x + (f_y(x_0,y_0)+\varepsilon_2)\Delta y$

Let $x(s) = x_0 + s u_1$, $y(s) = y_0 + s u_2$

$$\Delta x = x(s) - x(0) = s u_1 \qquad \Delta s = s - 0$$
$$\Delta y = y(s) - y(0) = s u_2$$

$$\Rightarrow D_{\vec{u}} f(x_0,y_0) = \lim_{\Delta s\to 0} \frac{\Delta z}{\Delta s}$$
$$= \lim_{\Delta s\to 0}\left((f_x(x_0,y_0)+0)\frac{\Delta x}{\Delta s} + (f_y(x_0,y_0)+0)\frac{\Delta y}{\Delta s}\right)$$
$$= f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2 = \nabla f(x_0,y_0)\cdot\vec{u}$$

Eg 2 $f(x,y) = x^2 + xy$

Note: f is differentiable

let $\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$D_{\vec{u}} f(1,2) = ?$

<u>Sol</u> $= \nabla f(1,2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$= \frac{f_x(1,2)}{\sqrt{2}} + \frac{f_y(1,2)}{\sqrt{2}}$

$f_x = 2x + y, \ f_y = x$

$\text{Ans} = \frac{4+1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$

Eg3 $f(x,y) = \begin{cases} \dfrac{4xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Evaluate $D_{\vec{u}} f(0,0)$ and $\nabla f(0,0) \cdot \vec{u}$

<u>Ans</u>: $f_x(0,0) = \lim_{x \to 0} \dfrac{f(x,0) - f(0,0)}{x - 0} = 0$

$f_y(0,0) = \lim_{x \to 0} \dfrac{f(0,y) - f(0,0)}{y - 0} = 0$

$$D_{\vec{u}} f(0,0) = \lim_{s \to 0} \frac{f(su_1, su_2) - f(0,0)}{s - 0} \quad \left(\underline{\underline{u_1^2 + u_2^2 = 1}} \right)$$

$$= \lim_{s \to 0} \frac{\frac{4s^3 u_1 u_2}{s^2}}{s - 0} = 4u_1u_2 \neq \underbrace{\nabla f(0,0)}_{=(0,0)} \cdot \vec{u}$$

Eg 4 $f(x,y) = ax + by + c$

$\nabla f(0,0)\,\vec{u} \overset{?}{=} D_{\vec{u}} f(0,0)$

Sol $f_x(0,0) = a$, $f_y(0,0) = b$

$$D_{\vec{u}} f(0,0) = \lim_{s \to 0} \frac{(asu_1 + bsu_2 + c) - c}{s}$$

$$= au_1 + bu_2 = \nabla f(0,0) \cdot \vec{u}$$

Remark. f is diff. at (x_0, y_0)

$\Rightarrow$ $z = f(x,y)$ has a tangent plane at $(x_0, y_0, f(x_0, y_0))$

$\Rightarrow$ f is close to a linear function

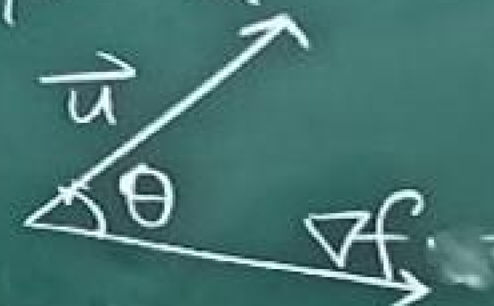
$\Rightarrow$ $D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

Properties of ∇f

If f is diff. at (x_0, y_0)

$$\Rightarrow D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

$$= |\nabla f(x_0, y_0)| \cdot |\vec{u}| \cos\theta$$



$\Rightarrow$ (1) f increase (decrease) most rapidly in the direction of $\vec{u}$ if $\cos\theta = 1$ (-1)

i.e. in the direction of ∇f $(-\nabla f)$

(2) $D_{\vec{u}} f(x_0, y_0) = 0 \Leftrightarrow \nabla f(x_0, y_0) \perp \vec{u}$

(3) Another interpretation of ∇f

level curves of f $=\{(x,y)\,|\,f(x,y)=c\}$

$\Rightarrow \nabla f \perp$ level curve.

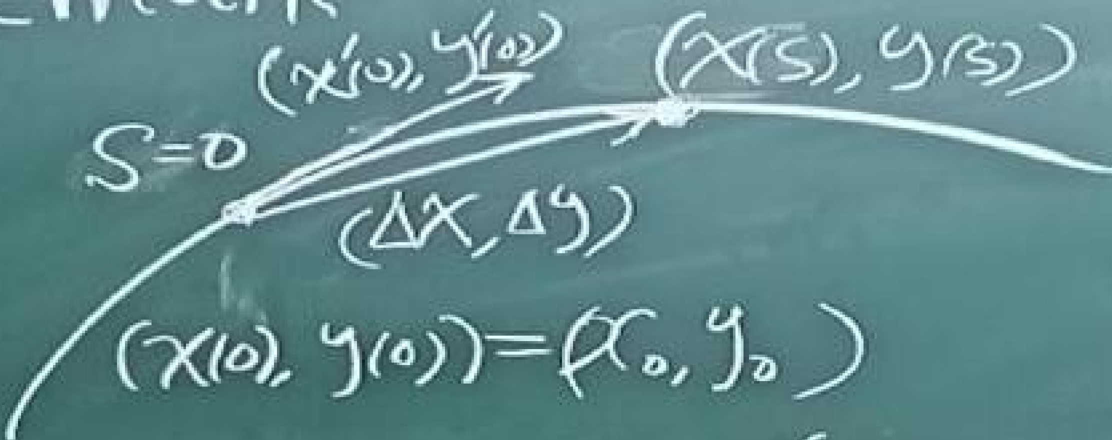
pf: Let $(x(s),y(s))$ be a level curve of f (differentiable) with $(x(0),y(0))=(x_0,y_0)$

$\Rightarrow f(x(s),y(s))=\text{constant}$

$\frac{d}{ds}\Big|_{s=0} \Rightarrow \nabla f(x_0,y_0)\cdot(x'(0),y'(0))=0$

$(x'(0),y'(0))$ = a tangent vector of level curve

Remark



$$\begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix} = \lim_{s \to 0} \begin{pmatrix} \dfrac{x(s) - x(0)}{s-0} \\ \dfrac{y(s)-y(0)}{s-0} \end{pmatrix}$$

$$= \lim_{\Delta s \to 0} \frac{1}{\Delta s} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad \Delta s = s - 0$$

$=$ a tangent vector

$\therefore (x'(0), y'(0))$ points in the direction of tangent line at $(x(0), y(0))$

Eg1 Find the tangent line (normal) of $\frac{x^2}{4} + y^2 = 2$ at $(-2, 1)$

Sol Let $f(x,y) = \frac{x^2}{4} + y^2$

$\because \frac{x^2}{4} + y^2 = 2$ is a level curve of f

$\therefore \nabla f(-2,1) =$ normal vector of the level curve at $(-2,1)$

tangent line:

$$(x+2,\ y-1) \cdot \underset{\underset{(-1,\,2)}{\parallel}}{\nabla f(-2,1)} = 0$$

Normal line:

$$\frac{y-1}{x-2} = \frac{2}{-1}$$

Algebraic Rule for gradient

(i) $\nabla(f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla(c f(x,y)) = c \nabla f(x,y)$

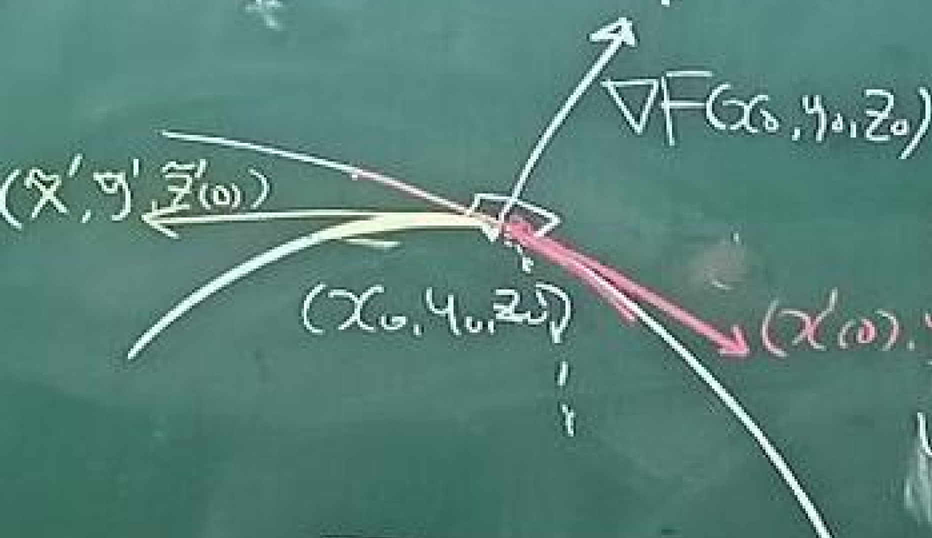
(iii) $\nabla(f \cdot g) = f \nabla g + g \nabla f$

(iv) $\nabla\left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2}$

Pf: check x-component and y-component directly

Tangent plane and normal line of level surface of $F(x,y,z)$ (diff.)

: $\{(x,y,z) \mid F(x,y,z) = C\}$



Suppose $\{x(t), y(t), z(t)\}$

$\subset$ {level surface of F} at (x_0,y_0,z_0)

with $(x(0),y(0),z(0)) = (x_0,y_0,z_0)$

$$\Rightarrow F(x(t),y(t),z(t)) = \text{constant} = F(x_0,y_0,z_0)$$

$$\left.\frac{d}{dt}\right|_{t=0} \Rightarrow \nabla F(x_0,y_0,z_0) \cdot (\underbrace{x'(0), y'(0), z'(0)}_{\text{a tangent vector of the level surface at } (x_0,y_0,z_0)}) = 0$$

For any curve $(x(t), y(t), z(t))$
on the surface passing (x_0, y_0, z_0)

$\nabla F(x_0, y_0, z_0) \perp (x'(0), y'(0), z'(0))$

$\Rightarrow \nabla F(x_0, y_0, z_0) \perp$ any tangent vector at (x_0, y_0, z_0)

$\Rightarrow \nabla F(x_0, y_0, z_0) \perp$ tangent plane at (x_0, y_0, z_0)

tangent plane

$$(x-x_0, y-y_0, z-z_0) \cdot \nabla F(x_0, y_0, z_0) = 0$$

normal line: $\dfrac{x-x_0}{F_x(x_0,y_0,z_0)} = \dfrac{y-y_0}{F_y(x_0,y_0,z_0)} = \dfrac{z-z_0}{F_z(x_0,y_0,z_0)}$

or
$$\begin{cases} x(t) = x_0 + F_x(x_0, y_0, z_0)t \\ y(t) = y_0 + F_y(x_0, y_0, z_0)t \\ z(t) = z_0 + F_z(x_0, y_0, z_0)t \end{cases}$$