

# Chain Rule

Single variable function

$$\frac{d}{dx} f(g(x)) = \frac{df}{dy} \Big|_{y=g(x)} \cdot \frac{dg(x)}{dx}$$

Two variables function  $z = f(x, y)$

$$\frac{d}{dt} f(x(t), y(t)) = \lim_{\substack{t \rightarrow t_0 \\ \Delta t \rightarrow 0}} \frac{\Delta z}{\Delta t}$$

$$\left( \Delta z = f(x(t), y(t)) - f(x(t_0), y(t_0)) \right)$$

(assume  $f$  is differentiable)

$$= \lim_{\substack{\Delta t \rightarrow 0 \\ t \rightarrow t_0}} \left( f_x(x(t), y(t)) + \varepsilon_1 \right) \frac{\Delta x}{\Delta t} + \left( f_y(x(t), y(t)) + \varepsilon_2 \right) \frac{\Delta y}{\Delta t}$$

(\*)

When  $\Delta t \rightarrow 0$  ( $t \rightarrow t_0$ )

we have  $(x(t), y(t)) \Rightarrow (x(t_0), y(t_0))$

(assume  $x(t), y(t)$  are also  
differentiable at  $t_0$ ,  
therefore continuous at  $t_0$ )

$$\Rightarrow (\Delta x, \Delta y) \rightarrow (0, 0)$$

$$\Rightarrow \lim_{\Delta t \rightarrow 0} \varepsilon_1 = 0 = \lim_{\Delta t \rightarrow 0} \varepsilon_2$$

$\therefore (*)$

$$\begin{aligned} &= f_x(x(t_0), y(t_0)) \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} \\ &+ f_y(x(t_0), y(t_0)) \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} \\ &= f_x(x(t_0), y(t_0)) x'(t_0) + f_y(x(t_0), y(t_0)) y'(t_0) \end{aligned}$$

In other words

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t)$$

Similarly,  $\partial_s f(x(s,t), y(s,t), z(s,t))$

$$= f_x(x(s,t), y(s,t), z(s,t)) \cdot \partial_s x(s,t)$$

$$+ f_y(x(s,t), y(s,t), z(s,t)) \cdot \partial_s y(s,t)$$

$$+ f_z(x(s,t), y(s,t), z(s,t)) \cdot \partial_s z(s,t)$$

$\partial_t f(x(s,t), y(s,t), z(s,t))$

$$= f_x(x(s,t), y(s,t), z(s,t)) \cdot \partial_t x(s,t)$$

$$+ f_y(x(s,t), y(s,t), z(s,t)) \cdot \partial_t y(s,t)$$

$$+ f_z(x(s,t), y(s,t), z(s,t)) \cdot \partial_t z(s,t)$$

# Implicit differentiation revisited

Find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  if  $F(x, y, z) = 0$

(i.e. If  $z(x, y)$  is implicitly defined  
by  $F(x, y, z) = 0$ , find  $\frac{\partial z}{\partial x}$ ,  $\frac{\partial z}{\partial y}$ )

Sol  $F(x, y, z(x, y)) = 0$  (as two functions of  $(x, y)$ )  
 $\Rightarrow \partial_x F(x, y, z(x, y)) = 0$

$$\Rightarrow \partial_1 F \cdot \frac{\partial x}{\partial x} + \partial_2 F \cdot \frac{\partial y}{\partial x} + \partial_3 F \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \partial_1 F(x, y, z(x, y)) + \partial_3 F(x, y, z(x, y)) \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} \text{ at } (x, y, z) = - \frac{\partial_1 F(x, y, z)}{\partial_3 F(x, y, z)} = - \frac{\partial_x F(x, y, z)}{\partial_z F(x, y, z)}$$

Similarly

$$\frac{\partial z}{\partial y} \text{ at } (x, y, z) = - \frac{\partial_x F(x, y, z)}{\partial_z F(x, y, z)}$$

Ex 1 Find  $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$  at (1, 1, 1)

from  $F(x, y, z) = xy + z^3x - 2yz = 0$

$$\text{Ans. } \partial_x F(1, 1, 1) = y + z^3 \Big|_{(1, 1, 1)} = 2$$

$$\partial_y F(1, 1, 1) = x - 2z \Big|_{(1, 1, 1)} = -1$$

$$\partial_z F(1, 1, 1) = 3z^2x - 2y \Big|_{(1, 1, 1)} = 1$$

$$\therefore \frac{\partial z}{\partial x} \text{ at } (1, 1, 1) = - \frac{2}{1} = -2$$

$$\frac{\partial z}{\partial y} \text{ at } (1, 1, 1) = - \frac{(-1)}{1} = 1$$

# Directional derivative

Def:  $\left(\frac{df}{ds}\right)_{\vec{u}, P_0} (= D_{\vec{u}} f(x_0, y_0))$   
( $P_0 = (x_0, y_0)$ )

Derivative of  $f(x, y)$  at  $P_0 = (x_0, y_0)$  in the direction of the unit vector  $\vec{u} = (u_1, u_2)$   
( $u_1^2 + u_2^2 = 1$ )

$\stackrel{\text{def}}{=} \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s - 0}$

$(= \lim_{s \rightarrow 0} \frac{f(x(s), y(s)) - f(x(0), y(0))}{s - 0})$

where  $x(s) = x_0 + su_1$ ,  $y(s) = y_0 + su_2$

Thm If  $f(x, y)$  is differentiable

at  $(x_0, y_0)$ , then  $D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

where  $\nabla f = (f_x, f_y)$  (gradient of  $f$ )

pf:  $\Delta z = (f_x(x_0, y_0) + \varepsilon_1) \Delta x + (f_y(x_0, y_0) + \varepsilon_2) \Delta y$

Let  $x(s) = x_0 + s u_1$ ,  $y(s) = y_0 + s u_2$

$$\Delta x = x(s) - x(0) = s u_1 \quad \Delta s = s - 0$$

$$\Delta y = y(s) - y(0) = s u_2$$

$$\Rightarrow D_{\vec{u}} f(x_0, y_0) = \lim_{\Delta s \rightarrow 0} \frac{\Delta z}{\Delta s}$$

$$= \lim_{\Delta s \rightarrow 0} \left( f_x(x_0, y_0) + 0 \right) \frac{\Delta x}{\Delta s} + \left( f_y(x_0, y_0) + 0 \right) \frac{\Delta y}{\Delta s}$$

$$= f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2 = \nabla f(x_0, y_0) \cdot \vec{u}$$

$$\text{Ex 2 } f(x, y) = x^2 + xy$$

Note:  $f$  is differentiable

$$\text{let } \vec{u} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$D_{\vec{u}} f(1, 2) = ?$$

$$\underline{\text{Sol}} = \nabla f(1, 2) \cdot \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$= \frac{f_x(1, 2)}{\sqrt{2}} + \frac{f_y(1, 2)}{\sqrt{2}}$$

$$f_x = 2x + y, \quad f_y = x$$

$$\text{Ans} = \frac{4+1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

$$\text{Ex 3 } f(x,y) = \begin{cases} \frac{4xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Evaluate  $D_{\vec{u}} f(0,0)$  and  $\nabla f(0,0) \cdot \vec{u}$

Ans:  $f_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x-0} = 0$

$$f_y(0,0) = \lim_{y \rightarrow 0} \frac{f(0,y) - f(0,0)}{y-0} = 0$$

$$D_{\vec{u}} f(0,0) = \lim_{s \rightarrow 0} \frac{f(su_1, su_2) - f(0,0)}{s-0} \quad (\underline{\underline{u_1^2 + u_2^2 = 1}})$$

$$= \lim_{s \rightarrow 0} \frac{4s^3 u_1 u_2}{s^2} = 4u_1 u_2 \neq \nabla f(0,0) \cdot \vec{u}$$

$\parallel$   
 $(0,0)$

$$\text{Ex 4 } f(x, y) = ax + by + c$$

$$\nabla f(0, 0) \cdot \vec{u} \neq D_{\vec{u}} f(0, 0)$$

$$\text{Sol } f_x(0, 0) = a, \quad f_y(0, 0) = b$$

$$D_{\vec{u}} f(0, 0) = \lim_{s \rightarrow 0} \frac{(as u_1 + bs u_2 + c) - c}{s}$$

$$= a u_1 + b u_2 = \nabla f(0, 0) \cdot \vec{u}$$

Remark.  $f$  is diff. at  $(x_0, y_0)$

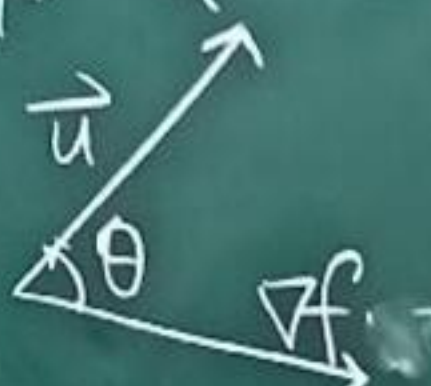
$\Rightarrow z = f(x, y)$  has a tangent plane at  $(x_0, y_0, f(x_0, y_0))$

$\Rightarrow f$  is close to a linear function

$$\Rightarrow D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

# Properties of $\nabla f$

If  $f$  is diff. at  $(x_0, y_0)$

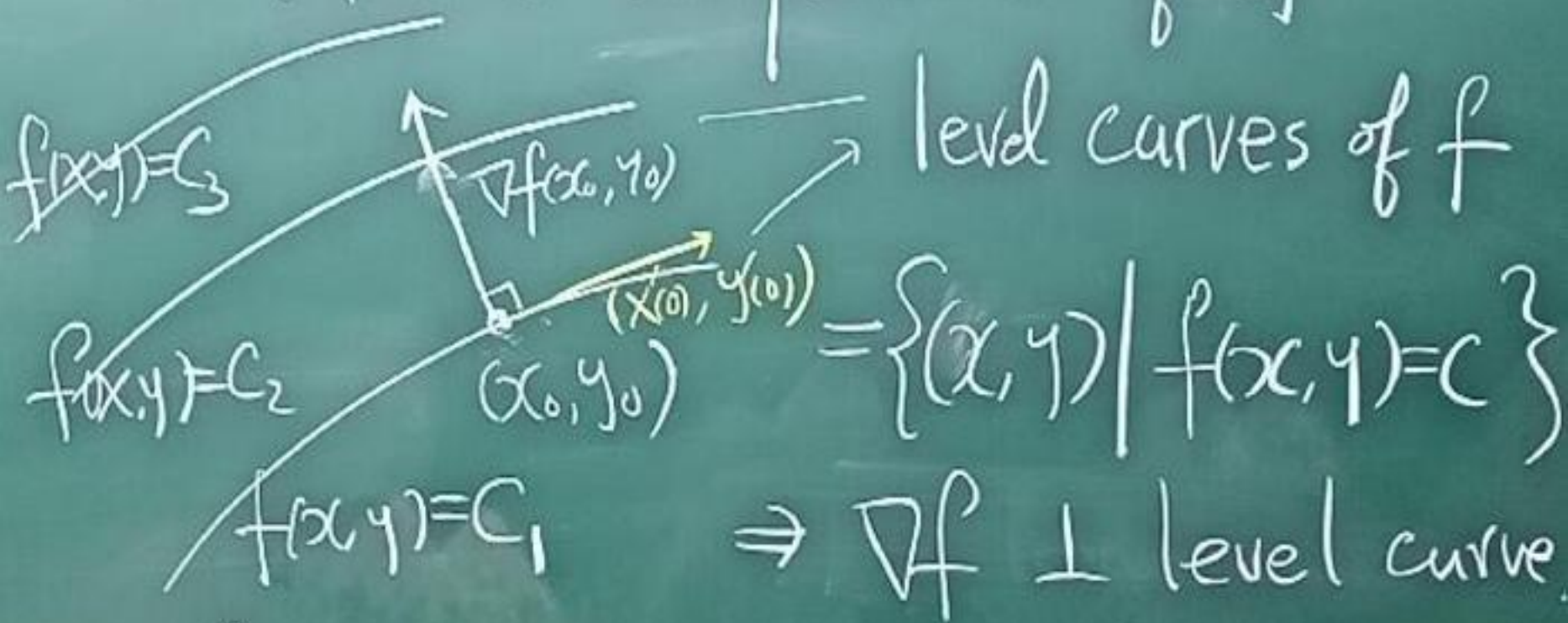
$$\begin{aligned}\Rightarrow D_{\vec{u}} f(x_0, y_0) &= \nabla f(x_0, y_0) \cdot \vec{u} \\ &= |\nabla f(x_0, y_0)| \cdot |\vec{u}| \cos \theta\end{aligned}$$


The diagram shows two vectors originating from the same point. One vector is labeled  $\vec{u}$  and the other is labeled  $\nabla f$ . The angle between them is labeled  $\theta$ .

$\Rightarrow$  (1)  $f$  <sup>increase</sup> most rapidly  
(decrease) in the direction of  $\vec{u}$  if  $\cos \theta = 1$   
 $(-1)$   
i.e. in the direction of  $\nabla f$   
 $(-\nabla f)$

$$(2) D_{\vec{u}} f(x_0, y_0) = 0 \iff \nabla f(x_0, y_0) \perp \vec{u}$$

(3) Another interpretation of  $\nabla f$



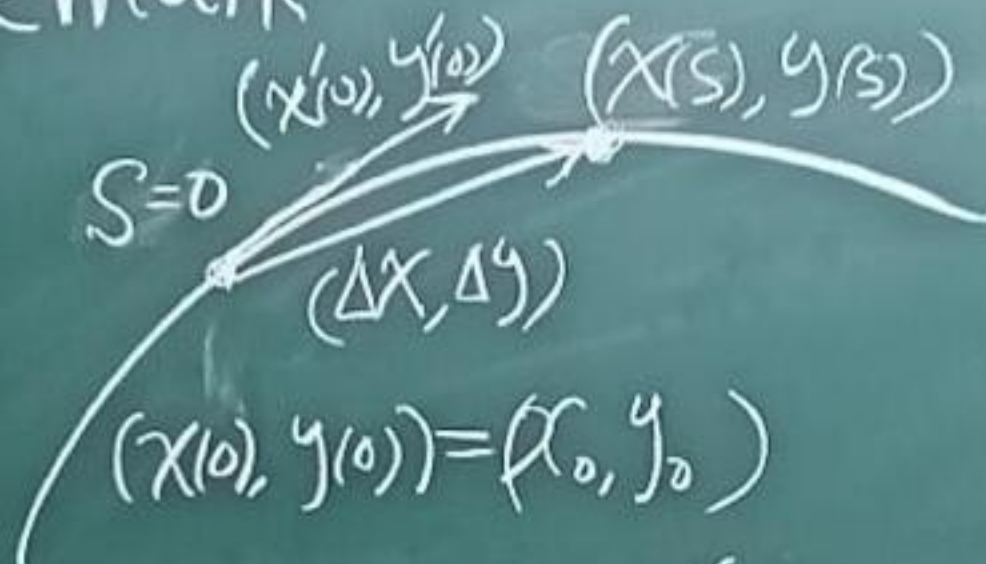
pf: Let  $(x(s), y(s))$  be a level curve of  $f$  (differentiable) with  $(x(0), y(0)) = (x_0, y_0)$

$$\Rightarrow f(x(s), y(s)) = \text{constant}$$

$$\frac{d}{ds} \Big|_{s=0} \Rightarrow \nabla f(x_0, y_0) \cdot (x'(0), y'(0)) = 0$$

$(x'(0), y'(0)) =$  a tangent vector of level curve

Remark



$$\begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix} = \lim_{s \rightarrow 0} \begin{pmatrix} \frac{x(s) - x(0)}{s - 0} \\ \frac{y(s) - y(0)}{s - 0} \end{pmatrix}$$

$$= \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \quad \Delta s = s - 0$$

= a tangent vector

$\therefore (x'(0), y'(0))$  points in the direction of tangent line at  $(x(0), y(0))$

Eg1 Find the tangent line  
(normal)

of  $\frac{x^2}{4} + y^2 = 2$  at  $(-2, 1)$

Sol Let  $f(x, y) = \frac{x^2}{4} + y^2$

$\therefore \frac{x^2}{4} + y^2 = 2$  is a level curve  
of  $f$

$\therefore \nabla f(-2, 1) =$  normal vector of  
the level curve at  $(-2, 1)$

tangent line:  $(-1, 2)$

$$(x+2, y-1) \cdot \nabla f(-2, 1) = 0$$

Normal line:

$$\frac{y-1}{x-2} = \frac{2}{-1}$$

# Algebraic Rule for gradient

$$(i) \nabla(f \pm g) = \nabla f \pm \nabla g$$

$$(ii) \nabla(c f(x, y)) = c \nabla f(x, y)$$

$$(iii) \nabla(f \cdot g) = f \nabla g + g \nabla f$$

$$(iv) \nabla\left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2}$$

$\square$ : Check  $x$ -component  
and  $y$ -component directly

Tangent plane and normal line  
of level surface of  $F(x, y, z)$  (diff.)

$$\{ (x, y, z) \mid F(x, y, z) = C \}$$

Suppose  $\{x(t), y(t), z(t)\}$   
C level surface of  $F$   
at  $(x_0, y_0, z_0)$   
with  $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$

$$\Rightarrow F(x(t), y(t), z(t)) = \text{constant} = F(x_0, y_0, z_0)$$

$$\frac{d}{dt} \Big|_{t=0} \Rightarrow \nabla F(x_0, y_0, z_0) \cdot \underbrace{(x'(0), y'(0), z'(0))}_{\text{a tangent vector}} = 0$$

a tangent vector  
of the level surface at  $(x_0, y_0, z_0)$

For any curve  $(x(t), y(t), z(t))$

on the surface passing  $(x_0, y_0, z_0)$

$$\nabla F(x_0, y_0, z_0) \perp (x'(0), y'(0), z'(0))$$

$$\Rightarrow \nabla F(x_0, y_0, z_0) \perp \begin{array}{l} \text{any tangent} \\ \text{vector at } (x_0, y_0, z_0) \end{array}$$

$$\Rightarrow \nabla F(x_0, y_0, z_0) \perp \begin{array}{l} \text{tangent plane at} \\ (x_0, y_0, z_0) \end{array}$$

tangent plane

$$(x-x_0, y-y_0, z-z_0) \cdot \nabla F(x_0, y_0, z_0) = 0$$

$$\text{normal line: } \frac{x-x_0}{F_x(x_0, y_0, z_0)} = \frac{y-y_0}{F_y(x_0, y_0, z_0)} = \frac{z-z_0}{F_z(x_0, y_0, z_0)}$$

$$\text{or } \begin{cases} x(t) = x_0 + F_x(x_0, y_0, z_0)t \\ y(t) = y_0 + F_y(x_0, y_0, z_0)t \\ z(t) = z_0 + F_z(x_0, y_0, z_0)t \end{cases}$$