

Thm I If $L(x) = a(x - x_0) + b$
and $E(x) = f(x) - L(x)$

satisfies (i) $E(x_0) = 0$

(ii) $\lim_{x \rightarrow x_0} \frac{E(x)}{x - x_0} = 0$

Then (1) $f(x)$ is diff. at $x = x_0$.

(2) $a = f'(x_0)$, $b = f(x_0)$

$$f(x) - L(x) = (x - x_0) \varepsilon$$

Rm: (ii) \Leftrightarrow

$$\lim_{x \rightarrow x_0} \varepsilon = 0$$

pf of Thm I

$$(i) \Rightarrow b = f(x_0)$$

$$(ii) \Rightarrow 0 = \lim_{x \rightarrow x_0} \frac{f(x) - (f(x_0) + a(x - x_0))}{x - x_0}$$

$$\Rightarrow \left(\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \right) - a = 0$$

\Rightarrow " limit exists and $= a$

Thm II (See Remark_on_definition_of_differentiability_v03.pdf)

If $L(x, y) = a(x - x_0) + b(y - y_0) + c$

and $E(x, y) = f(x, y) - L(x, y)$

Satisfies (i) $E(x_0, y_0) = 0$

$$(ii) \lim_{(x, y) \rightarrow (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

(Then, we say $f(x, y)$ is diff at (x_0, y_0))

Then (1) $\partial_x f(x_0, y_0), \partial_y f(x_0, y_0)$ exist

$$(2) a = \partial_x f(x_0, y_0), b = \partial_y f(x_0, y_0), c = f(x_0, y_0)$$

$$\text{Rm: (ii) } \Leftrightarrow \begin{cases} f(x, y) - L(x, y) = \varepsilon \sqrt{(x - x_0)^2 + (y - y_0)^2} \\ \lim_{(x, y) \rightarrow (x_0, y_0)} \varepsilon = 0 \end{cases}$$

Def $z = f(x, y)$ is differentiable at (x_0, y_0) if

(i) $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.

(ii) $f(x, y) = L(x, y) + \varepsilon_1(x - x_0) + \varepsilon_2(y - y_0)$
(or $= L(x, y) + \underline{\varepsilon} \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}$)

where

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

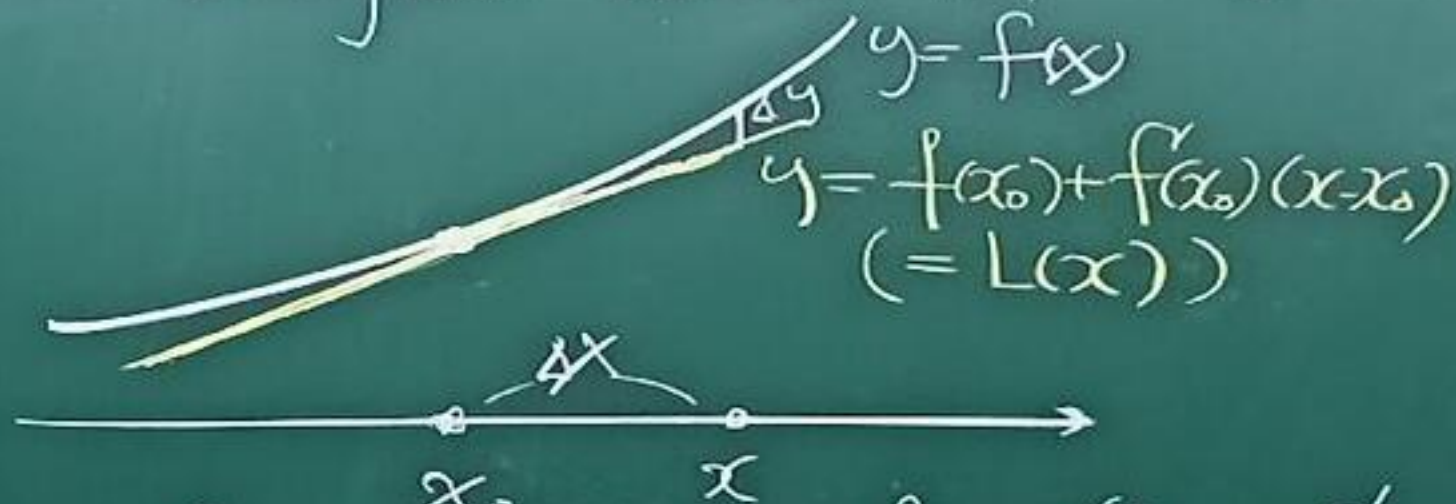
where $\lim_{(x, y) \rightarrow (x_0, y_0)} \varepsilon_1, \varepsilon_2, \underline{\varepsilon} = 0$

i.e. $z = f(x, y)$ and $z = L(x, y)$ are

tangent at $(x_0, y_0, f(x_0, y_0))$

i.e. $\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$ ($= \lim_{(x, y) \rightarrow (x_0, y_0)} \underline{\varepsilon}$)

Remark differentiability and tangent line in 1D



$$\lim_{x \rightarrow x_0} \frac{f(x) - L(x)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{f(x) - (f(x_0) + f'(x_0)(x - x_0))}{x - x_0}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \quad \parallel \quad 0$$

Rm Two curves $\begin{cases} y = f(x) \\ y = g(x) \end{cases}$ are tangent at (x_0, y_0)

if (i) $f(x_0) = g(x_0) = y_0$

(ii) $\lim_{x \rightarrow x_0} \frac{f(x) - g(x)}{x - x_0} = 0$ ($|f(x) - g(x)|$ is smaller than $|x - x_0|$)

Rm (homework)

$$\varepsilon_1(x-x_0) + \varepsilon_2(y-y_0) = \varepsilon \cdot \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

Eg 6. Are $\begin{cases} y=f_1(x)=e^x \\ y=f_2(x)=1+x \end{cases}$ tangent at $(0,1)$?

Sol. (i) $f_1(0)=1, f_2(0)=1$

(ii) $\lim_{x \rightarrow 0} \frac{f_1(x) - f_2(x)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x} = 0$ (Yes)

Eg 7. Are $\begin{cases} z=f_1(x,y)=x^2+y^2 \\ z=f_2(x,y)=0 \end{cases}$ tangent at $(0,0,0)$?

Sol. (i) $f_1(0,0)=0, f_2(0,0)=0$

(ii) $\lim_{(x,y) \rightarrow (0,0)} \frac{f_1(x,y) - f_2(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = 0$ (Yes)

In definition of $f(x,y)$ is differentiable at (x_0, y_0)

$$L(x,y) \stackrel{\text{def}}{=} f(x_0, y_0) + f'_x(x_0, y_0)(x-x_0) + f'_y(x_0, y_0)(y-y_0)$$

$$f(x,y) = L(x,y) + \varepsilon_1(x-x_0) + \varepsilon_2(y-y_0) \quad (1)$$

$$\Leftrightarrow f(x,y) = L(x,y) + \varepsilon \sqrt{(x-x_0)^2 + (y-y_0)^2} \quad (2)$$

$$\Leftrightarrow \Delta z = f'_x(x_0, y_0)\Delta x + f'_y(x_0, y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \quad (3)$$

$$\Leftrightarrow \Delta z = f'_x(x_0, y_0)\Delta x + f'_y(x_0, y_0)\Delta y + \varepsilon \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad (4)$$

where $\Delta z = f(x,y) - f(x_0, y_0)$, $\Delta x = x - x_0$, $\Delta y = y - y_0$

and $\lim_{(x,y) \rightarrow (x_0, y_0)} (\varepsilon_1, \varepsilon_2, \varepsilon) = (0, 0, 0)$ (1)-(4) are all the same

The textbook uses (3)

Def

1st order approximation to each other

$z = f(x, y)$ and $z = g(x, y)$ are tangent at (x_0, y_0, z_0)

if (i) $f(x_0, y_0) = g(x_0, y_0) = z_0$

$$(ii) \lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - g(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

$\therefore f$ is differentiable at (x_0, y_0)

$\Leftrightarrow z = f(x, y)$ and $z = L(x, y)$ are tangent at $(x_0, y_0, f(x_0, y_0))$

In fact, it can be shown that, if $z = f(x, y)$ has a tangent plane at (x_0, y_0, z_0) , then $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ must exist, and the tangent plane must be

$$z = L(x, y)$$

(See Supplement)

Thm 2: If $f, f_x, f_y, f_{xy}, f_{yx}$ are all cont. in an open region R and $(x_0, y_0) \in R$

$$\begin{aligned} \text{Then } f_{xy}(x_0, y_0) &= f_{yx}(x_0, y_0) \\ &= \partial_y \partial_x f(x_0, y_0) \quad \partial_x \partial_y f(x_0, y_0) \end{aligned}$$

(R is an open region if R has no boundary point)

Note:

$f_{xy}(x_0, y_0), f_{yx}(x_0, y_0)$ both exist

~~\Rightarrow~~ $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$

$$\text{Eg 1 } f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Then $f_{xy} = f_{yx}$ on $\mathbb{R}^2 - (0, 0)$ (direct computation)

How about $f_{xy}(0, 0) \stackrel{?}{=} f_{yx}(0, 0)$

Sol $f_{xy}(0, 0) = \lim_{y \rightarrow 0} \frac{f_x(0, y) - f_x(0, 0)}{y - 0}$

$$f_x(0, y) = \partial_x \left(xy \frac{x^2 - y^2}{x^2 + y^2} \right) \Big|_{(0, y)} = \dots$$

$$f_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x \cdot 0 \frac{x^2 - 0}{x^2 + 0} - 0}{x} = 0$$

Similarly for $f_{yx}(0, 0)$ (homework)

Thm 3: R is an open region
 $(x_0, y_0) \in R$. If f, f_x, f_y
are all defined in R
and continuous at (x_0, y_0) ,
then f is differentiable
at (x_0, y_0) .

Prf: See Appendix 9.

Ex 2 $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$

Is $f(x, y)$ cont at $(0, 0)$?
differentiable at $(0, 0)$?

Ans (i) $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) \neq 0$ (yes)

(ii) Step 1: find $L(x, y)$

$f_x(0, 0) = 0 = f_y(0, 0) \Rightarrow L(x, y) = 0$
(exercise)

Step 2 $\lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - 0}{\sqrt{x^2+y^2}} \neq 0$

$= \lim_{(x, y) \rightarrow (0, 0)} \frac{xy}{x^2+y^2}$ does not exist Ans. NO

Thm 4 $f(x, y)$ is diff. at (x_0, y_0) (1)

$\Rightarrow f(x, y)$ is cont. at (x_0, y_0) (2)

pf. (1) $\Leftrightarrow \Delta z = f'_x(x_0, y_0) \Delta x + f'_y(x_0, y_0) \Delta y$
 $+ \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$
 $= (f'_x(x_0, y_0) + \varepsilon_1) \Delta x + (f'_y(x_0, y_0) + \varepsilon_2) \Delta y$

$\Rightarrow \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \Delta z = 0$

$\Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) - f(x_0, y_0) = 0$

$\Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0) \Leftrightarrow (2)$