Thm I If $L(x) = a(x-x_0)+b$

and $E(x) = f(x) - L(x)$

Satisfies (i) $E(x_0) = 0$

(ii) $\lim_{x \to x_0} \frac{E(x)}{x-x_0} = 0$

Then (1) $f(x)$ is diff. at $x=x_0$

(2) $a = f'(x_0)$, $b = f(x_0)$

$$f(x) - L(x) = (x-x_0)\varepsilon$$

Rm: (ii) $\Leftrightarrow$ $\lim_{x \to x_0} \varepsilon = 0$

Pf of Thm I

(i) $\Longrightarrow b = f(x_0)$

(ii) $\Rightarrow 0 = \lim\limits_{x \to x_0} \dfrac{f(x) - (f(x_0) + a(x - x_0))}{x - x_0}$

$$\Longrightarrow \underbrace{\left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right)}_{\text{limit exists and } = a} - a = 0$$

$\Longrightarrow$ limit exists and $= a$

Thm II (See Remark_on_definition_of_differentiability_v03.pdf)

If $L(x,y) = a(x-x_0) + b(y-y_0) + c$

and $E(x) = f(x,y) - L(x,y)$

Satisfies (i) $E(x_0, y_0) = 0$

(ii) $\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$

(Then, we say $f(x,y)$ is diff at (x_0,y_0))

Then (1) $\partial_x f(x_0,y_0)$, $\partial_y f(x_0,y_0)$ exist

(2) $a = \partial_x f(x_0,y_0)$, $b = \partial_y f(x_0,y_0)$, $c = f(x_0,y_0)$

Rm: (ii) $\Longleftrightarrow \begin{cases} f(x,y) - L(x,y) = \varepsilon \sqrt{(x-x_0)^2+(y-y_0)^2} \\ \lim_{(x,y)\to(x_0,y_0)} \varepsilon = 0 \end{cases}$

Def $z = f(x,y)$ is differentiable at (x_0, y_0) if

(i) $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.

(ii) $f(x,y) = L(x,y) + \varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0)$

$\left(\text{or} = L(x,y) + \underline{\varepsilon} \cdot \sqrt{(x-x_0)^2 + (y-y_0)^2}\right)$

where

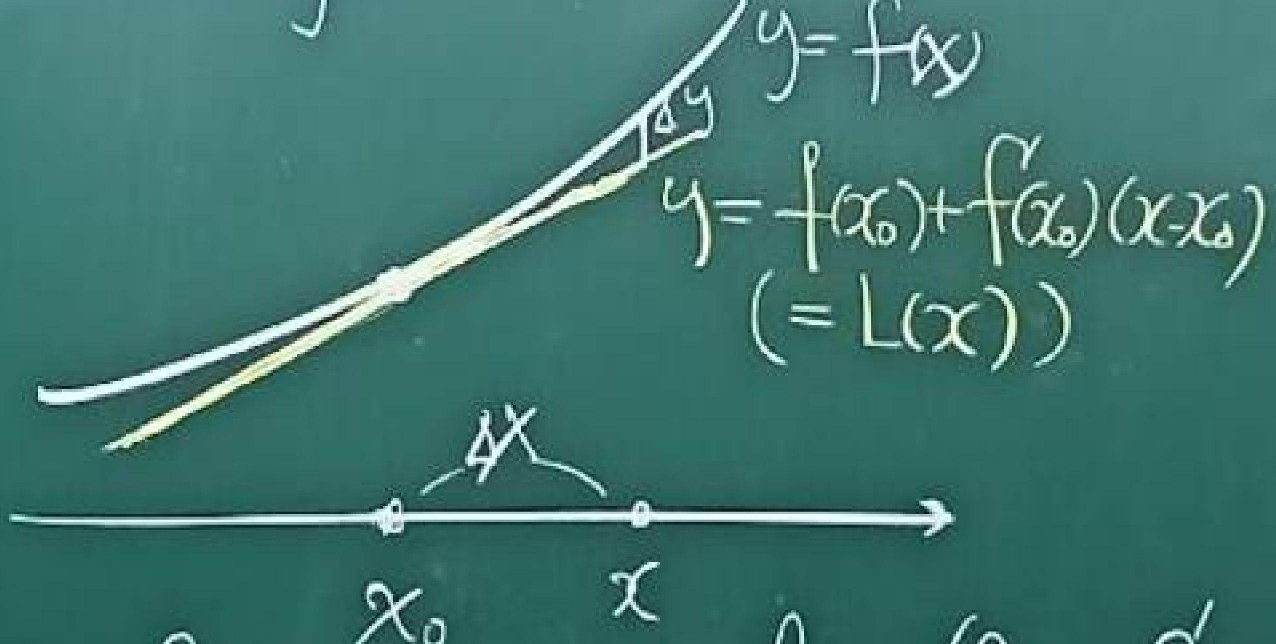
$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

where $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \varepsilon_1, \varepsilon_2, \underline{\varepsilon} = 0$

i.e. $z = f(x,y)$ and $z = L(x,y)$ are tangent at $(x_0, y_0, f(x_0,y_0))$

i.e. $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0 \left(= \lim_{(x,y)\to(x_0,y_0)} \underline{\varepsilon} \right)$$

Remark differentiability and tangent line in 1D



$$\lim_{x\to x_0}\frac{f(x)-L(x)}{x-x_0}=\lim_{x\to x_0}\frac{f(x)-(f(x_0)+f'(x_0)(x-x_0))}{x-x_0}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} \qquad \overset{\parallel}{0}$$

Rm Two curve $\begin{cases} y=f(x) \\ y=g(x)\end{cases}$ are tangent at (x_0,y_0)

if (i) $f(x_0)=g(x_0)=y_0$

(ii) $\lim_{x\to x_0}\frac{f(x)-g(x)}{x-x_0}=0$ $\left(\begin{array}{c}|f(x)-g(x)| \text{ is} \\ \text{smaller than } |x-x_0|\end{array}\right)$

Rm (homework)

$$\varepsilon_1(x-x_0)+\varepsilon_2(y-y_0)=\varepsilon\cdot\sqrt{(x-x_0)^2+(y-y_0)^2}$$

Eg 6. Are $\begin{cases} y=f_1(x)=e^x \\ y=f_2(x)=1+x \end{cases}$ tangent at $(0,1)$?

Sol: (i) $f_1(0)=1,\ f_2(0)=1$

(ii) $\displaystyle\lim_{x\to 0}\frac{f_1(x)-f_2(x)}{x-0}=\lim_{x\to 0}\frac{e^x-1-x}{x}=0$ (Yes)

Eg 7: Are $\begin{cases} z=f_1(x,y)=x^2+y^2 \\ z=f_2(x,y)=0 \end{cases}$ tangent at $(0,0,0)$?

Sol: (i) $f_1(0,0)=0,\ f_2(0,0)=0$

(ii) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{f_1(x,y)-f_2(x,y)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{x^2+y^2}}=0$ (Yes)

In definition of $f(x,y)$ is differentiable at (x_0,y_0)

$$L(x,y) \overset{\text{def}}{=} f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$f(x,y) = L(x,y) + \varepsilon_1(x-x_0) + \varepsilon_2(y-y_0) \qquad (1)$$

$$\Leftrightarrow f(x,y) = L(x,y) + \varepsilon\cdot\sqrt{(x-x_0)^2+(y-y_0)^2} \qquad (2)$$

$$\Leftrightarrow \Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \qquad (3)$$

$$\Leftrightarrow \Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon\cdot\sqrt{(\Delta x)^2+(\Delta y)^2} \qquad (4)$$

where $\Delta z = f(x,y) - f(x_0,y_0)$, $\Delta x = x - x_0$, $\Delta y = y - y_0$

and $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (\varepsilon_1, \varepsilon_2, \varepsilon) = (0,0,0)$

(1)-(4) are all the same

The textbook uses (3)

Def: 1st order approximation to each other

$=$

$z=f(x,y)$, $z=g(x,y)$ are tangent at (x_0,y_0,z_0)

if (i) $f(x_0,y_0)=g(x_0,y_0)=z_0$

(ii) $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-g(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$$

$\therefore$ f is differentiable at (x_0,y_0)

$\Longleftrightarrow$ $z=f(x,y)$ and $z=L(x,y)$ are tangent at $(x_0,y_0,f(x_0,y_0))$

In fact, it can be shown that, if $z = f(x,y)$ has a tangent plane at (x_0, y_0, z_0), then $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ must exist, and the tangent plane must be

$$z = L(x,y)$$

(See ~~Supplement~~)

Remark_on_definition_of_differentiability_v03.pdf

Thm 2: If $f, f_x, f_y, f_{xy}, f_{yx}$ are all cont. in an open region R and $(x_0, y_0) \in R$

Then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$

$$= \partial_y \partial_x f(x_0, y_0) \qquad \partial_x \partial_y f(x_0, y_0)$$

(R is an open region if R has no boundary point)

Note:

$f_{xy}(x_0, y_0)$, $f_{yx}(x_0, y_0)$ both exist

$$\not\Rightarrow f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

Eg1 $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

Then $f_{xy}=f_{yx}$ on $\mathbb{R}^2\setminus(0,0)$ $\left(\begin{array}{l}\text{direct}\\ \text{computation}\end{array}\right)$

How about $f_{xy}(0,0) \overset{?}{=} f_{yx}(0,0)$

<u>Sol</u> $f_{xy}(0,0)=\lim\limits_{y\to 0}\dfrac{f_x(0,y)-f_x(0,0)}{y-0}$

$f_x(0,y)=\partial_x\left(xy\dfrac{x^2-y^2}{x^2+y^2}\right)\Big|_{(0,y)}=\cdots$

$f_x(0,0)=\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x-0}=\lim\limits_{x\to 0}\dfrac{x\cdot 0\dfrac{x^2-0}{x^2+0}-0}{x}=0$

Similarly for $f_{yx}(0,0)$ (homework)

Thm 3: R is an open region $(x_0, y_0) \in R$. If f, f_x, f_y are all defined in R and continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

Pf: See Appendix 9.

Eg2 $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

Is $f(x,y)$ cont at $(0,0)$?

differentiable at $(0,0)$?

Ans (i) $\lim\limits_{(x,y)\to(0,0)} f(x,y) \overset{?}{=} 0$ (yes)

(ii) Step 1. find $L(x,y)$

$f_x(0,0)=0=f_y(0,0) \Rightarrow L(x,y)=0$

(exercise)

Step 2 $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y)-0}{\sqrt{x^2+y^2}} \overset{?}{=} 0$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist Ans. NO

Thm 4 $f(x,y)$ is diff. at (x_0,y_0) (1)

$\Rightarrow f(x,y)$ is cont. at (x_0,y_0) (2)

pf: (1) $\Leftrightarrow \Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$$= (f_x(x_0,y_0)+\varepsilon_1)\Delta x + (f_y(x_0,y_0)+\varepsilon_2)\Delta y$$

$$\Rightarrow \lim_{(\Delta x,\Delta y)\to(0,0)} \Delta z = 0$$

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) - f(x_0,y_0) = 0$$

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0) \Leftrightarrow (2)$$