Thm I If $L(x) = a(x-x_0)+b$

and $E(x) = f(x) - L(x)$

satisfies (i) $E(x_0) = 0$

(ii) $\lim\limits_{x \to x_0} \frac{E(x)}{x-x_0} = 0$

Then (1) $f(x)$ is diff. at $x = x_0$

(2) $a = f'(x_0)$, $b = f(x_0)$

$f(x) - L(x) = (x-x_0)\varepsilon$

Rm: (ii) <=> $\lim\limits_{x \to x_0} \varepsilon = 0$

Pf of Thm I

(i) $\Rightarrow b = f(x_0)$

(ii) $\Rightarrow 0 = \lim_{x \to x_0} \dfrac{f(x) - (f(x_0) + a(x - x_0))}{x - x_0}$

$\Rightarrow \left(\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} \right) - a = 0$

$\Rightarrow$ limit exists and $= a$

Thm II

If $L(x,y) = a(x-x_0) + b(y-y_0) + c$

and $E(x) = f(x,y) - L(x,y)$

Satisfies (i) $E(x_0, y_0) = 0$

(ii) $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$

(Then, we say $f(x,y)$ is diff at (x_0,y_0))

Then (1) $\partial_x f(x_0,y_0)$, $\partial_y f(x_0,y_0)$ exist

(2) $a = \partial_x f(x_0,y_0)$, $b = \partial_y f(x_0,y_0)$. $c = f(x_0,y_0)$

<u>Def</u> $z = f(x,y)$ is differentiable at (x_0, y_0) if

(i) $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.

(ii) $f(x,y) = L(x,y) + \varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0)$

$\left(\text{or} = L(x,y) + \underline{\varepsilon} \cdot \sqrt{(x-x_0)^2 + (y-y_0)^2} \right)$

where

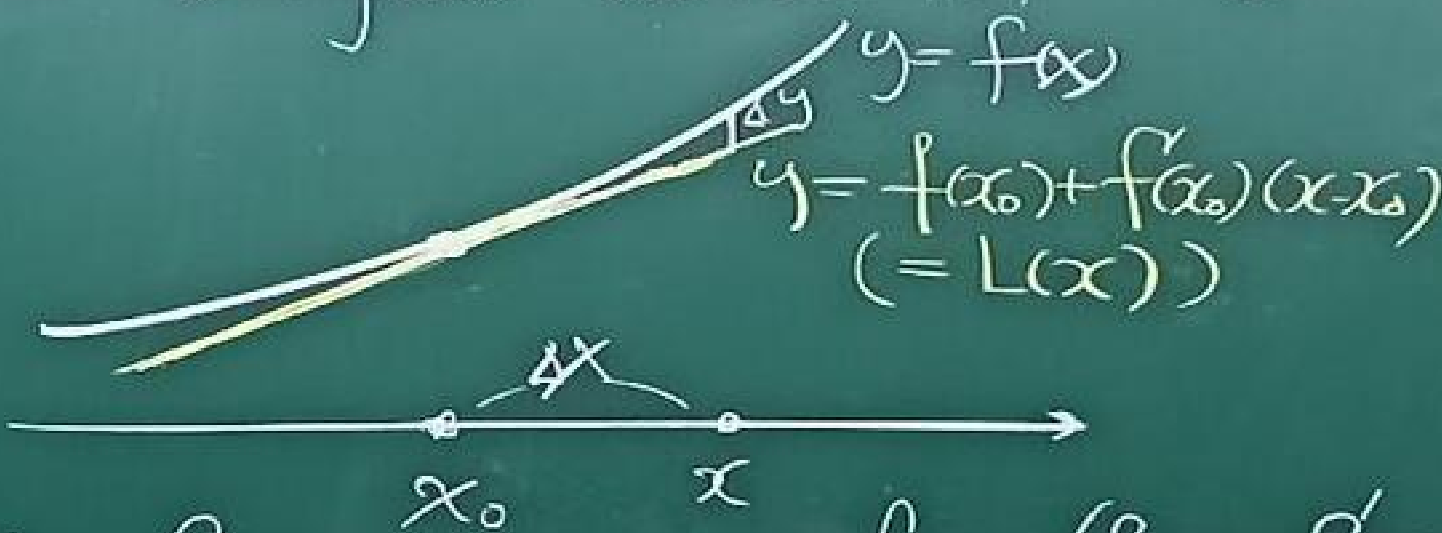
$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

where $\lim\limits_{(x,y)\to(x_0,y_0)} \varepsilon_1, \varepsilon_2, \underline{\varepsilon} = 0$

i.e. $f(x,y)$ and $L(x,y)$ are tangent at $(x_0, y_0, f(x_0, y_0))$

i.e. $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y) - L(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0 \left(= \lim\limits_{(x,y)\to(x_0,y_0)} \underline{\varepsilon} \right)$

Remark differentiability and tangent line in 1D



$$\lim_{x\to x_0}\frac{f(x)-L(x)}{x-x_0}=\lim_{x\to x_0}\frac{f(x)-(f(x_0)+f'(x_0)(x-x_0))}{x-x_0}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} \qquad \| \quad 0$$

Rm Two curve $\begin{cases} y=f(x) \\ y=g(x)\end{cases}$ are tangent at (x_0,y_0)

if (i) $f(x_0)=g(x_0)=y_0$

(ii) $\lim_{x\to x_0}\frac{f(x)-g(x)}{x-x_0}=0$ $\left(\begin{array}{c}|f(x)-g(x)| \text{ is} \\ \text{smaller than } |x-x_0|\end{array}\right)$

Rm (homework)

$$\varepsilon_1(x-x_0)+\varepsilon_2(y-y_0)=\varepsilon\cdot\sqrt{(x-x_0)^2+(y-y_0)^2}$$

Eg 6. Are $\begin{cases} y=f_1(x)=e^x \\ y=f_2(x)=1+x \end{cases}$ tangent at $(0,1)$?

Sol: (i) $f_1(0)=1$, $f_2(0)=1$

(ii) $\displaystyle\lim_{x\to 0}\frac{f_1(x)-f_2(x)}{x-0}=\lim_{x\to 0}\frac{e^x-1-x}{x}=0$ (Yes)

Eg 7: Are $\begin{cases} z=f_1(x,y)=x^2+y^2 \\ z=f_2(x,y)=0 \end{cases}$ tangent at $(0,0,0)$?

Sol: (i) $f_1(0,0)=0$, $f_2(0,0)=0$

(ii) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{f_1(x,y)-f_2(x,y)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{x^2+y^2}}=0$ (Yes)

In definition of $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$

$$L(x,y) \overset{\text{def}}{=} f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$f(x,y) = L(x,y) + \varepsilon_1(x-x_0) + \varepsilon_2(y-y_0) \qquad (1)$$

$$\Leftrightarrow f(x,y) = L(x,y) + \varepsilon\cdot\sqrt{(x-x_0)^2+(y-y_0)^2} \qquad (2)$$

$$\Leftrightarrow \Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \qquad (3)$$

$$\Leftrightarrow \Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon\cdot\sqrt{(\Delta x)^2+(\Delta y)^2} \qquad (4)$$

where $\Delta z = f(x,y) - f(x_0,y_0)$, $\Delta x = x - x_0$, $\Delta y = y - y_0$

and $\lim_{(x,y)\to(x_0,y_0)} (\varepsilon_1, \varepsilon_2, \varepsilon) = (0,0,0)$

(1)-(4) are all the same

The textbook uses (3)

Def.

$z=f(x,y)$, $z=g(x,y)$ are tangent at (x_0,y_0,z_0)

if (i) $f(x_0,y_0)=g(x_0,y_0)=z_0$

(ii) $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-g(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$$

$\therefore$ f is differetiable at (x_0,y_0)

$\Longleftrightarrow$ $z=f(x,y)$ and $z=L(x,y)$ are tangent at $(x_0,y_0,f(x_0,y_0))$

In fact, it can be shown that, if $z = f(x, y)$ has a tangent plane at (x_0, y_0, z_0), then $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ must exist, and the tangent plane must be

$$z = L(x, y)$$

(See supplement)

Thm 2: If $f, f_x, f_y, f_{xy}, f_{yx}$ are all cont. in an open region R and $(x_0, y_0) \in R$

Then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$

$= \partial_y \partial_x f(x_0, y_0) \qquad \qquad \overset{\shortparallel}{\partial_x \partial_y f(x_0, y_0)}$

(R is an open region if R has no boundary point)

Note:

$f_{xy}(x_0, y_0), f_{yx}(x_0, y_0)$ both exist

$\not\Rightarrow f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$

Eg 1 $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

Then $f_{xy}=f_{yx}$ on $\mathbb{R}^2\setminus(0,0)$ $\left(\begin{array}{c}\text{direct}\\\text{computation}\end{array}\right)$

How about $f_{xy}(0,0)\overset{?}{=}f_{yx}(0,0)$

<u>Sol</u> $f_{xy}(0,0)=\lim\limits_{y\to 0}\dfrac{f_x(0,y)-f_x(0,0)}{y-0}$

$f_x(0,y)=\partial_x\left(xy\dfrac{x^2-y^2}{x^2+y^2}\right)\Big|_{(0,y)}=\cdots$

$f_x(0,0)=\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x-0}=\lim\limits_{x\to 0}\dfrac{x\cdot 0\,\frac{x^2-0}{x^2+0}-0}{x}=0$

Similarly for $f_{yx}(0,0)$ (homework)

Thm 3: R is an open region $(x_0, y_0) \in R$. If f, f_x, f_y are all defined in R and continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

<u>pf</u>: See Appendix 9.

Eg2 $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Is $f(x,y)$ cont at $(0,0)$?

differentiable at $(0,0)$?

Ans (i) $\lim_{(x,y)\to(0,0)} f(x,y) \stackrel{?}{=} 0$ (yes)

(ii) Step 1. find $L(x,y)$

$f_x(0,0) = 0 = f_y(0,0) \Rightarrow L(x,y) = 0$

(exercise)

Step 2 $\lim_{(x,y)\to(0,0)} \dfrac{f(x,y)-0}{\sqrt{x^2+y^2}} \stackrel{?}{=} 0$

$= \lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist

Ans. NO

Thm 4 $f(x,y)$ is diff. at (x_0,y_0) (1)

$\Rightarrow f(x,y)$ is cont. at (x_0,y_0) (2)

pf: (1) $\Leftrightarrow \Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$$= (f_x(x_0,y_0)+\varepsilon_1)\Delta x + (f_y(x_0,y_0)+\varepsilon_2)\Delta y$$

$$\Rightarrow \lim_{(\Delta x,\Delta y)\to(0,0)} \Delta z = 0$$

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) - f(x_0,y_0) = 0$$

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0) \Leftrightarrow (2)$$

Chain Rule

Single variable function

$$\frac{d}{dx} f(g(x)) = \left.\frac{df}{dy}\right|_{y=g(x)} \cdot \frac{dg(x)}{dx}$$

Two variables function $z = f(x, y)$

$$\frac{d}{dt} f(x(t), y(t)) = \lim_{\substack{t \to t_0 \\ \Delta t = t - t_0}} \frac{\Delta z}{\Delta t}$$

$$\left(\Delta z = f(x(t), y(t)) - f(x(t_0), y(t_0)) \right)$$

(assume f is differentiable)

$$= \lim_{\substack{\Delta t \to 0 \\ (t \to t_0)}} \left(f_x(x(t), y(t)) + \varepsilon_1\right) \frac{\Delta x}{\Delta t} + \left(f_y(x(t), y(t)) + \varepsilon_2\right) \frac{\Delta y}{\Delta t}$$

$(*)$

When $\Delta t \to 0$ $(t \to t_0)$

we have $(x(t), y(t)) \Rightarrow (x(t_0), y(t_0))$

(assume $x(t), y(t)$ are also differentiable at t_0, therefore continuous at t_0)

$\Rightarrow (\Delta x, \Delta y) \longrightarrow (0, 0)$

$\Rightarrow \lim\limits_{\Delta t \to 0} \varepsilon_1 = 0 = \lim\limits_{\Delta t \to 0} \varepsilon_2$

$\therefore$ $(*)$

$= f_x(x(t_0), y(t_0)) \lim\limits_{\Delta t \to 0} \dfrac{\Delta x}{\Delta t}$

$+ f_y(x(t_0), y(t_0)) \lim\limits_{\Delta t} \dfrac{\Delta y}{\Delta t}$

$= f_x(x(t_0), y(t_0))\, x'(t_0) + f_y(x(t_0), y(t_0))\, y'(t_0)$

In other words

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t)$$

Similarly, $\partial_s f(x(s,t), y(s,t), z(s,t))$

$$= f_x(x(s,t), y(s,t), z(s,t)) \cdot \partial_s x(s,t)$$
$$+ f_y(\qquad\qquad) \cdot \partial_s y(s,t)$$
$$+ f_z(\qquad\qquad) \partial_s z(s,t)$$

$\partial_t f(x(s,t), y(s,t), z(s,t))$

$$= f_x(\quad \cdots \quad) \cdot \partial_t x(s,t)$$
$$+ f_y(\quad \cdots \quad) \partial_t y(s,t)$$
$$+ f_z(\qquad\qquad) \partial_t z(s,t)$$

Implicit differentiation revisited

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $F(x,y,z)=0$

(i.e. If $z(x,y)$ is implicitly defined by $F(x,y,z)=0$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$)

<u>Sol</u> $\quad F(x,y,z(x,y)) = 0 \quad \left(\begin{array}{c}\text{as two}\\ \text{functions}\\ \text{of } (x,y)\end{array}\right)$

$$\Rightarrow \partial_x F(x,y,z(x,y)) = 0$$

$$\Rightarrow \partial_1 F \cdot \underbrace{\frac{\partial x}{\partial x}}_{=1} + \partial_2 F \cdot \underbrace{\frac{\partial y}{\partial x}}_{=0} + \partial_3 F \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \partial_1 F(x,y,z(x,y)) + \partial_3 F(x,y,z(x,y)) \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} \text{ at } (x,y,z) = -\frac{\partial_1 F(x,y,z)}{\partial_3 F(x,y,z)} = -\frac{\partial_x F(x,y,z)}{\partial_z F(x,y,z)}$$

Similarly

$$\frac{\partial z}{\partial y} \text{ at } (x,y,z) = -\frac{\partial_y F(x,y,z)}{\partial_z F(x,y,z)}$$

Eg 1 Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(1,1,1)$

from $F(x,y,z) = xy + z^3x - 2yz = 0$

Ans. $\partial_x F(1,1,1) = y + z^3 \big|_{(1,1,1)} = 2$

$\partial_y F(1,1,1) = x - 2z \big|_{(1,1,1)} = -1$

$\partial_z F(1,1,1) = 3z^2x - 2y \big|_{(1,1,1)} = 1$

$\therefore \frac{\partial z}{\partial x}$ at $(1,1,1) = -\frac{2}{1} = -2$

$\frac{\partial z}{\partial y}$ at $(1,1,1) = -\frac{(-1)}{1} = 1$

Directional derivative

Def: $\left(\frac{df}{ds}\right)_{\vec{u},P_0}$ $\left(= D_{\vec{u}} f(x_0, y_0)\right)$

$(P_0 = (x_0, y_0))$

$\left(\begin{array}{l}\text{Derivative of } f(x,y) \text{ at} \\ P_0 = (x_0, y_0) \text{ in the direction} \\ \text{of the } \underline{\text{unit vector}} \ \vec{u} = (u_1, u_2) \\ \qquad (u_1^2 + u_2^2 = 1)\end{array}\right)$

$$\overset{\text{def}}{=\!=} \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s - 0}$$

$$\left(= \lim_{s \to 0} \frac{f(x(s), y(s)) - f(x(0), y(0))}{s - 0} \right.$$

$$\left. \text{where } x(s) = x_0 + su_1,\ y(s) = y_0 + su_2 \right)$$

Thm If $f(x,y)$ is differentiable at (x_0,y_0), then $D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \vec{u}$

where $\nabla f = (f_x, f_y)$ (gradient of f)

pf: $\Delta z = (f_x(x_0,y_0)+\varepsilon_1)\Delta x + (f_y(x_0,y_0)+\varepsilon_2)\Delta y$

Let $x(s) = x_0 + su_1$, $y(s) = y_0 + su_2$

$\Delta x = x(s) - x(0) = su_1$

$\Delta y = y(s) - y(0) = su_2$

$\Delta s = s - 0$

$$\Rightarrow D_{\vec{u}} f(x_0,y_0) = \lim_{\Delta s \to 0} \frac{\Delta z}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \left((f_x(x_0,y_0)+0)\frac{\Delta x}{\Delta s} + (f_y(x_0,y_0)+0)\frac{\Delta y}{\Delta s} \right)$$

$$= f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2 = \nabla f(x_0,y_0) \cdot \vec{u}$$

Eg 2 $f(x,y) = x^2 + xy$

Note: f is differentiable

let $\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$D_{\vec{u}} f(1,2) = ?$

Sol $= \nabla f(1,2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$= \frac{f_x(1,2)}{\sqrt{2}} + \frac{f_y(1,2)}{\sqrt{2}}$

$f_x = 2x + y, \quad f_y = x$

$Ans = \frac{4+1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$

Eg3 $f(x,y) = \begin{cases} \dfrac{4xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Evaluate $D_{\vec{u}} f(0,0)$ and $\nabla f(0,0) \cdot \vec{u}$

Ans: $f_x(0,0) = \lim\limits_{x \to 0} \dfrac{f(x,0) - f(0,0)}{x-0} = 0$

$f_y(0,0) = \lim\limits_{x \to 0} \dfrac{f(0,y) - f(0,0)}{y-0} = 0$

$$D_{\vec{u}} f(0,0) = \lim_{s \to 0} \frac{f(su_1, su_2) - f(0,0)}{s-0} \quad (u_1^2 + u_2^2 = 1)$$

$$= \lim_{s \to 0} \frac{\frac{4s^3 u_1 u_2}{s^2}}{s-0} = 4u_1u_2 \neq \underbrace{\nabla f(0,0)}_{(0,0)} \cdot \vec{u}$$

Eg 4 $f(x,y) = ax + by + c$

$\nabla f(0,0)\,\vec{u} \overset{?}{=} D_{\vec{u}} f(0,0)$

<u>Sol</u> $f_x(0,0) = a, \quad f_y(0,0) = b$

$$D_{\vec{u}} f(0,0) = \lim_{s \to 0} \frac{(asu_1 + bsu_2 + c) - c}{s}$$

$$= au_1 + bu_2 = \nabla f(0,0) \cdot \vec{u}$$

<u>Remark</u>. f is diff. at (x_0, y_0)

$\Rightarrow$ $z = f(x,y)$ has a tangent plane at $(x_0, y_0, f(x_0, y_0))$

$\Rightarrow$ f is close to a linear function

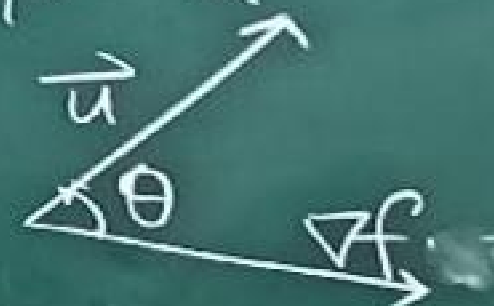
$\Rightarrow$ $D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

Properties of ∇f

If f is diff. at (x_0, y_0)

$\Rightarrow D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

$= |\nabla f(x_0, y_0)| \cdot |\vec{u}| \cos\theta$



$\Rightarrow$ (1) f increase (decrease) most rapidly

in the direction of $\vec{u}$ if $\cos\theta = 1$ (-1)

i.e. in the direction of ∇f $(-\nabla f)$

(2) $D_{\vec{u}} f(x_0, y_0) = 0 \Leftrightarrow \nabla f(x_0, y_0) \perp \vec{u}$

(3) Another interpretation of ∇f

level curves of f $=\{(x,y)\mid f(x,y)=c\}$

$\Rightarrow \nabla f \perp$ level curve.

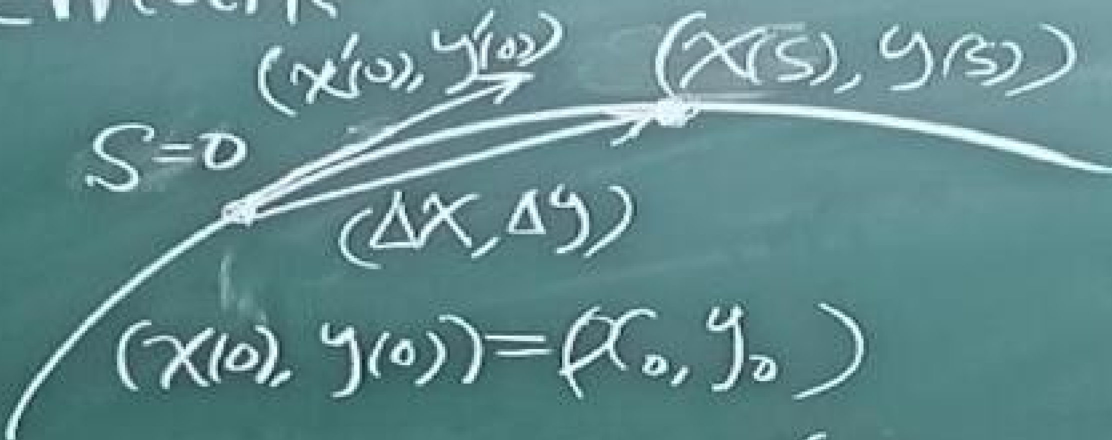
pf: Let $(x(s),y(s))$ be a level curve of f (differentiable) with $(x(0),y(0))=(x_0,y_0)$

$\Rightarrow f(x(s),y(s))=\text{constant}$

$\left.\frac{d}{ds}\right|_{s=0} \Rightarrow \nabla f(x_0,y_0)\cdot(x'(0),y'(0))=0$

$(x'(0),y'(0))$ = a tangent vector of level curve

Remark



$$\begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix} = \lim_{s\to 0} \begin{pmatrix} \frac{x(s)-x(0)}{s-0} \\ \frac{y(s)-y(0)}{s-0} \end{pmatrix}$$

$$= \lim_{\Delta s \to 0} \frac{1}{\Delta s} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad \Delta s = s - 0$$

$=$ a tangent vector

$\therefore (x'(0), y'(0))$ points in the direction of tangent line at $(x(0), y(0))$

Eg1 Find the tangent line (normal) of $\frac{x^2}{4}+y^2=2$ at $(-2,1)$

Sol Let $f(x,y)=\frac{x^2}{4}+y^2$

$\therefore \frac{x^2}{4}+y^2=2$ is a level curve of f

$\therefore \nabla f(-2,1)=$ normal vector of the level curve at $(-2,1)$

tangent line:

$$(x+2,\ y-1)\cdot \underset{(-1,\ 2)}{\underbrace{\nabla f(-2,1)}}=0$$

Normal line:

$$\frac{y-1}{x-2}=\frac{2}{-1}$$

Algebraic Rule for gradient

(i) $\nabla(f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla(c f(x,y)) = c \nabla f(x,y)$

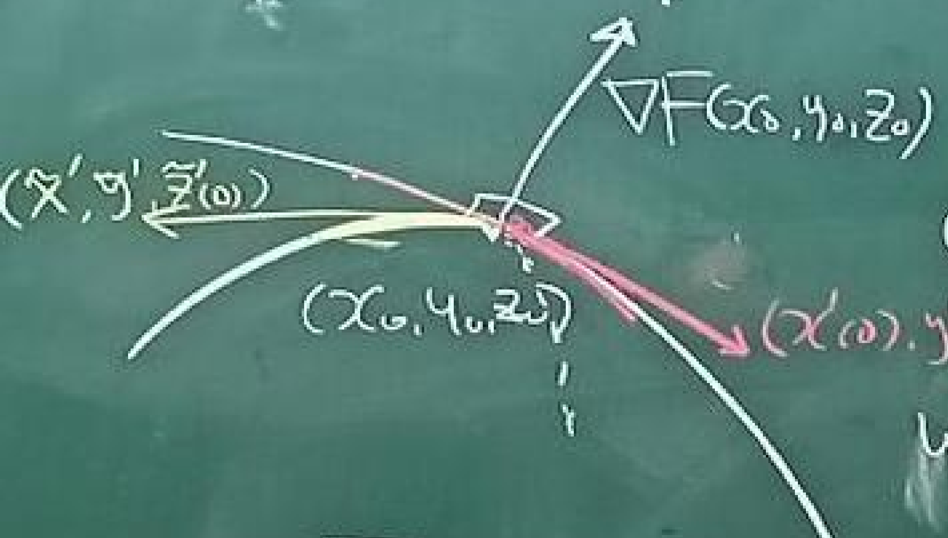
(iii) $\nabla(f \cdot g) = f \nabla g + g \nabla f$

(iv) $\nabla\left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2}$

証: check x-component and y-component directly

Tangent plane and normal line of level surface of $F(x,y,z)$ (diff.)

: $\{(x,y,z) \mid F(x,y,z)=C\}$



Suppose $\{x(t), y(t), z(t)\} \subset \{\text{level surface of } F\}$ at (x_0,y_0,z_0)

with $(x(0),y(0),z(0))=(x_0,y_0,z_0)$

$$\Rightarrow F(x(t),y(t),z(t)) = \text{constant} = F(x_0,y_0,z_0)$$

$$\left.\frac{d}{dt}\right|_{t=0} \Rightarrow \nabla F(x_0,y_0,z_0)\cdot\underbrace{(x'(0),y'(0),z'(0))}_{\substack{\text{a tangent vector}\\ \text{of the level surface at } (x_0,y_0,z_0)}} = 0$$

For any curve $(x(t), y(t), z(t))$
on the surface passing (x_0, y_0, z_0)

$\nabla F(x_0, y_0, z_0) \perp (x'(0), y'(0), z'(0))$

$\Rightarrow \nabla F(x_0, y_0, z_0) \perp$ any tangent vector at (x_0, y_0, z_0)

$\Rightarrow \nabla F(x_0, y_0, z_0) \perp$ tangent plane at (x_0, y_0, z_0)

tangent plane

$$(x - x_0, y - y_0, z - z_0) \cdot \nabla F(x_0, y_0, z_0) = 0$$

normal line: $\dfrac{x - x_0}{F_x(x_0, y_0, z_0)} = \dfrac{y - y_0}{F_y(x_0, y_0, z_0)} = \dfrac{z - z_0}{F_z(x_0, y_0, z_0)}$

or $$\begin{cases} x(t) = x_0 + F_x(x_0, y_0, z_0)t \\ y(t) = y_0 + F_y(x_0, y_0, z_0)t \\ z(t) = z_0 + F_z(x_0, y_0, z_0)t \end{cases}$$