

Thm I If  $L(x) = a(x - x_0) + b$   
and  $E(x) = f(x) - L(x)$

satisfies (i)  $E(x_0) = 0$

(ii)  $\lim_{x \rightarrow x_0} \frac{E(x)}{x - x_0} = 0$

Then (1)  $f(x)$  is diff. at  $x = x_0$

(2)  $a = f'(x_0)$ ,  $b = f(x_0)$

$$f(x) - L(x) = (x - x_0)\epsilon$$

Rm: (ii)  $\Leftrightarrow \lim_{x \rightarrow x_0} \epsilon = 0$

pf of Thm I

$$(i) \Rightarrow b = f'(x_0)$$

$$(ii) \Rightarrow 0 = \lim_{x \rightarrow x_0} \frac{f(x) - (f(x_0) + a(x - x_0))}{x - x_0}$$

$$\Rightarrow \left( \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \right) - a = 0$$

$\Rightarrow$  " " limit exists and  $= a$

## Thm II

If  $L(x, y) = a(x - x_0) + b(y - y_0) + c$

and  $E(x, y) = f(x, y) - L(x, y)$

Satisfies (i)  $E(x_0, y_0) = 0$

$$(ii) \lim_{(x, y) \rightarrow (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

(Then, we say  $f(x, y)$  is diff at  $(x_0, y_0)$ )

Then (1)  $\partial_x f(x_0, y_0), \partial_y f(x_0, y_0)$  exist

$$(2) a = \partial_x f(x_0, y_0), b = \partial_y f(x_0, y_0), c = f(x_0, y_0)$$

Def  $z = f(x, y)$  is differentiable at  $(x_0, y_0)$  if

(i)  $f'_x(x_0, y_0)$  and  $f'_y(x_0, y_0)$  exist.

(ii)  $f(x, y) = L(x, y) + \varepsilon_1(x - x_0) + \varepsilon_2(y - y_0)$   
(or  $= L(x, y) + \underline{\varepsilon} \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}$ )

where

$$L(x, y) = f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0)$$

where  $\lim_{(x, y) \rightarrow (x_0, y_0)} \varepsilon_1, \varepsilon_2, \underline{\varepsilon} = 0$

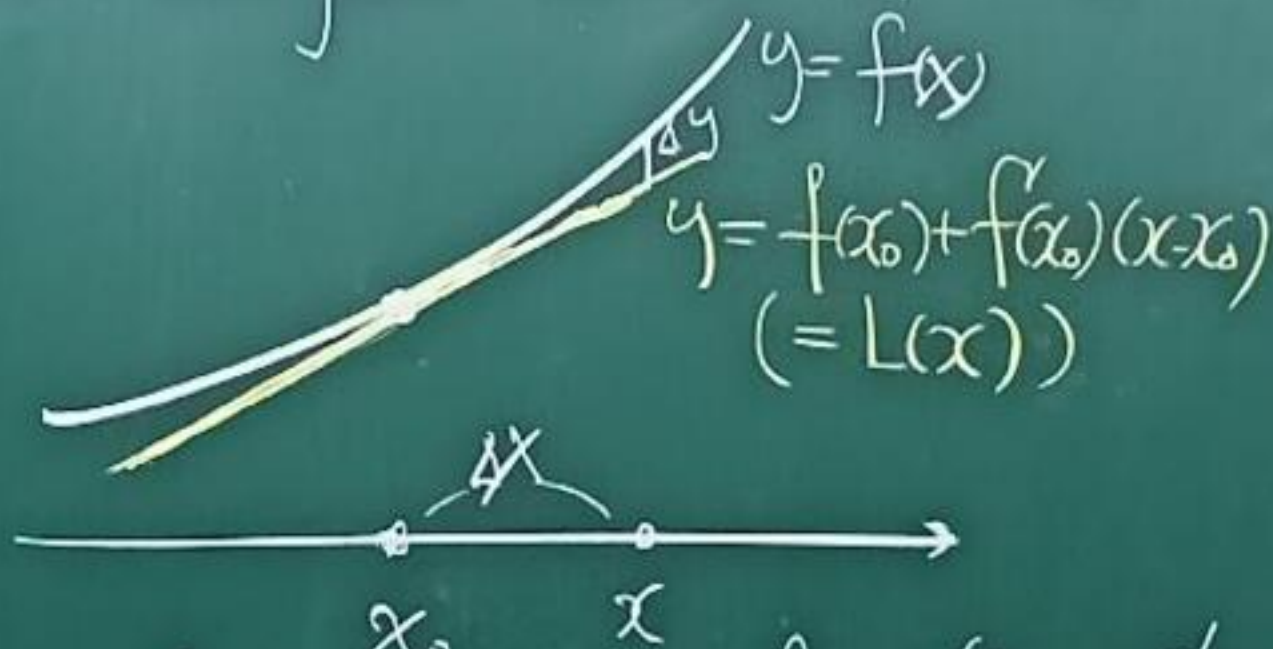
i.e.  $f(x, y)$  and  $L(x, y)$  are

tangent at  $(x_0, y_0, f(x_0, y_0))$

i.e.  $\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0 \quad (= \lim_{(x, y) \rightarrow (x_0, y_0)} \underline{\varepsilon})$

# Remark differentiability

and tangent line in 1D



$$\lim_{x \rightarrow x_0} \frac{f(x) - L(x)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{f(x) - (f(x_0) + f'(x_0)(x - x_0))}{x - x_0}$$
$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \quad \parallel \quad 0$$

Rm Two curves  $\begin{cases} y = f(x) \\ y = g(x) \end{cases}$  are tangent at  $(x_0, y_0)$

if (i)  $f(x_0) = g(x_0) = y_0$

(ii)  $\lim_{x \rightarrow x_0} \frac{f(x) - g(x)}{x - x_0} = 0$  ( $|f(x) - g(x)|$  is smaller than  $|x - x_0|$ )

## Rm (homework)

$$\varepsilon_1(x-x_0) + \varepsilon_2(y-y_0) = \varepsilon \cdot \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

Eg 6. Are  $\begin{cases} y=f_1(x)=e^x \\ y=f_2(x)=1+x \end{cases}$  tangent at  $(0,1)$ ?

Sol. (i)  $f_1(0)=1, f_2(0)=1$

(ii)  $\lim_{x \rightarrow 0} \frac{f_1(x) - f_2(x)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x} = 0$  (Yes)

Eg 7. Are  $\begin{cases} z=f_1(x,y)=x^2+y^2 \\ z=f_2(x,y)=0 \end{cases}$  tangent at  $(0,0,0)$ ?

Sol. (i)  $f_1(0,0)=0, f_2(0,0)=0$

(ii)  $\lim_{(x,y) \rightarrow (0,0)} \frac{f_1(x,y) - f_2(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = 0$  (Yes)

In definition of  $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y)$

$$L(x,y) \stackrel{\text{def}}{=} f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)$$

$$f(x,y) = L(x,y) + \varepsilon_1(x-x_0) + \varepsilon_2(y-y_0) \quad (1)$$

$$\Leftrightarrow f(x,y) = L(x,y) + \varepsilon \sqrt{(x-x_0)^2 + (y-y_0)^2} \quad (2)$$

$$\Leftrightarrow \Delta z = f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \quad (3)$$

$$\Leftrightarrow \Delta z = f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta y + \varepsilon \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad (4)$$

where  $\Delta z = f(x,y) - f(x_0,y_0)$ ,  $\Delta x = x - x_0$ ,  $\Delta y = y - y_0$

and  $\lim_{(x,y) \rightarrow (x_0,y_0)} (\varepsilon_1, \varepsilon_2, \varepsilon) = (0, 0, 0)$  (1)-(4) are all the same

The textbook uses (3)

Def.

$z = f(x, y)$  and  $z = g(x, y)$  are tangent at  $(x_0, y_0, z_0)$

if (i)  $f(x_0, y_0) = g(x_0, y_0) = z_0$

$$(ii) \lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - g(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

$\therefore f$  is differentiable at  $(x_0, y_0)$

$\Leftrightarrow z = f(x, y)$  and  $z = L(x, y)$  are tangent at  $(x_0, y_0, f(x_0, y_0))$

In fact, it can be shown that, if  $z = f(x, y)$  has a tangent plane at  $(x_0, y_0, z_0)$ , then  $f_x(x_0, y_0)$ ,  $f_y(x_0, y_0)$  must exist, and the tangent plane must be

$$z = L(x, y)$$

(See Supplement)

Thm 2: If  $f, f_x, f_y, f_{xy}, f_{yx}$   
are all cont. in an open region  $R$   
and  $(x_0, y_0) \in R$

$$\text{Then } f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0) \\ = \partial_y \partial_x f(x_0, y_0) \quad \partial_x \partial_y f(x_0, y_0)$$

( $R$  is an open region if  
 $R$  has no boundary point)

Note:

$f_{xy}(x_0, y_0), f_{yx}(x_0, y_0)$  both exist

$$\Rightarrow f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

$$\text{Eg 1 } f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Then  $f_{xy} = f_{yx}$  on  $\mathbb{R}^2 - (0, 0)$  (direct computation)

How about  $f_{xy}(0, 0) \stackrel{?}{=} f_{yx}(0, 0)$

Sol  $f_{xy}(0, 0) = \lim_{y \rightarrow 0} \frac{f_x(0, y) - f_x(0, 0)}{y - 0}$

$$f_x(0, y) = \partial_x \left( xy \frac{x^2 - y^2}{x^2 + y^2} \right) \Big|_{(0, y)} = \dots$$

$$f_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x \cdot 0 \frac{x^2 - 0}{x^2 + 0} - 0}{x} = 0$$

Similarly for  $f_{yx}(0, 0)$  (homework)

Thm 3:  $R$  is an open region  
 $(x_0, y_0) \in R$ . If  $f, f_x, f_y$   
are all defined in  $R$   
and continuous at  $(x_0, y_0)$ ,  
then  $f$  is differentiable  
at  $(x_0, y_0)$ .

Prf: See Appendix 9.

Ex 2  $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$

Is  $f(x, y)$  cont at  $(0, 0)$ ?  
differentiable at  $(0, 0)$ ?

Ans (i)  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) \neq 0$  (yes)

(ii) Step 1: find  $L(x, y)$

$f_x(0, 0) = 0 = f_y(0, 0) \Rightarrow L(x, y) = 0$   
(exercise)

Step 2  $\lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - 0}{\sqrt{x^2+y^2}} \neq 0$

$= \lim_{(x, y) \rightarrow (0, 0)} \frac{xy}{x^2+y^2}$  does not exist  
Ans. NO

Thm 4  $f(x, y)$  is diff. at  $(x_0, y_0)$  (1)

$\Rightarrow f(x, y)$  is cont. at  $(x_0, y_0)$  (2)

pf. (1)  $\Leftrightarrow \Delta z = f'_x(x_0, y_0)\Delta x + f'_y(x_0, y_0)\Delta y$   
 $+ \varepsilon_1\Delta x + \varepsilon_2\Delta y$

$$= (f'_x(x_0, y_0) + \varepsilon_1)\Delta x + (f'_y(x_0, y_0) + \varepsilon_2)\Delta y$$

$$\Rightarrow \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \Delta z = 0$$

$$\Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) - f(x_0, y_0)) = 0$$

$$\Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0) \Leftrightarrow (2)$$

# Chain Rule

Single variable function

$$\frac{d}{dx} f(g(x)) = \frac{df}{dy} \Big|_{y=g(x)} \cdot \frac{dg(x)}{dx}$$

Two variables function  $z = f(x, y)$

$$\frac{d}{dt} f(x(t), y(t)) = \lim_{\substack{t \rightarrow t_0 \\ \Delta t \rightarrow 0}} \frac{\Delta z}{\Delta t}$$

$$\left( \Delta z = f(x(t), y(t)) - f(x(t_0), y(t_0)) \right)$$

(assume  $f$  is differentiable)

$$= \lim_{\substack{\Delta t \rightarrow 0 \\ t \rightarrow t_0}} \left( f_x(x(t), y(t)) + \varepsilon_1 \right) \frac{\Delta x}{\Delta t} + \left( f_y(x(t), y(t)) + \varepsilon_2 \right) \frac{\Delta y}{\Delta t}$$

(\*)

When  $\Delta t \rightarrow 0$  ( $t \rightarrow t_0$ )

we have  $(x(t), y(t)) \Rightarrow (x(t_0), y(t_0))$

(assume  $x(t), y(t)$  are also  
differentiable at  $t_0$ ,  
therefore continuous at  $t_0$ )

$$\Rightarrow (\Delta x, \Delta y) \rightarrow (0, 0)$$

$$\Rightarrow \lim_{\Delta t \rightarrow 0} \varepsilon_1 = 0 = \lim_{\Delta t \rightarrow 0} \varepsilon_2$$

$\therefore (*)$

$$\begin{aligned} &= f_x(x(t_0), y(t_0)) \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} \\ &+ f_y(x(t_0), y(t_0)) \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} \\ &= f_x(x(t_0), y(t_0)) x'(t_0) + f_y(x(t_0), y(t_0)) y'(t_0) \end{aligned}$$

In other words

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t)$$

Similarly,  $\partial_s f(x(s,t), y(s,t), z(s,t))$

$$= f_x(x(s,t), y(s,t), z(s,t)) \cdot \partial_s x(s,t)$$

$$+ f_y(x(s,t), y(s,t), z(s,t)) \cdot \partial_s y(s,t)$$

$$+ f_z(x(s,t), y(s,t), z(s,t)) \cdot \partial_s z(s,t)$$

$$\partial_t f(x(s,t), y(s,t), z(s,t))$$

$$= f_x(x(s,t), y(s,t), z(s,t)) \cdot \partial_t x(s,t)$$

$$+ f_y(x(s,t), y(s,t), z(s,t)) \cdot \partial_t y(s,t)$$

$$+ f_z(x(s,t), y(s,t), z(s,t)) \cdot \partial_t z(s,t)$$

# Implicit differentiation revisited

Find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  if  $F(x, y, z) = 0$

(i.e. If  $z(x, y)$  is implicitly defined  
by  $F(x, y, z) = 0$ , find  $\frac{\partial z}{\partial x}$ ,  $\frac{\partial z}{\partial y}$ )

Sol  $F(x, y, z(x, y)) = 0$  (as two functions of  $(x, y)$ )  
 $\Rightarrow \partial_x F(x, y, z(x, y)) = 0$

$$\Rightarrow \partial_1 F \cdot \frac{\partial x}{\partial x} + \partial_2 F \cdot \frac{\partial y}{\partial x} + \partial_3 F \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \partial_1 F(x, y, z(x, y)) + \partial_3 F(x, y, z(x, y)) \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} \text{ at } (x, y, z) = - \frac{\partial_1 F(x, y, z)}{\partial_3 F(x, y, z)} = - \frac{\partial_x F(x, y, z)}{\partial_z F(x, y, z)}$$

Similarly

$$\frac{\partial z}{\partial y} \text{ at } (x, y, z) = - \frac{\partial_x F(x, y, z)}{\partial_z F(x, y, z)}$$

Ex 1 Find  $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$  at (1, 1, 1)

from  $F(x, y, z) = xy + z^3x - 2yz = 0$

$$\text{Ans. } \partial_x F(1, 1, 1) = y + z^3 \Big|_{(1, 1, 1)} = 2$$

$$\partial_y F(1, 1, 1) = x - 2z \Big|_{(1, 1, 1)} = -1$$

$$\partial_z F(1, 1, 1) = 3z^2x - 2y \Big|_{(1, 1, 1)} = 1$$

$$\therefore \frac{\partial z}{\partial x} \text{ at } (1, 1, 1) = - \frac{2}{1} = -2$$

$$\frac{\partial z}{\partial y} \text{ at } (1, 1, 1) = - \frac{(-1)}{1} = 1$$

# Directional derivative

Def:  $\left(\frac{df}{ds}\right)_{\vec{u}, P_0} (= D_{\vec{u}} f(x_0, y_0))$   
( $P_0 = (x_0, y_0)$ )

Derivative of  $f(x, y)$  at  $P_0 = (x_0, y_0)$  in the direction of the unit vector  $\vec{u} = (u_1, u_2)$   
( $u_1^2 + u_2^2 = 1$ )

$\stackrel{\text{def}}{=} \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s - 0}$

$(= \lim_{s \rightarrow 0} \frac{f(x(s), y(s)) - f(x(0), y(0))}{s - 0})$

where  $x(s) = x_0 + su_1$ ,  $y(s) = y_0 + su_2$

Thm If  $f(x, y)$  is differentiable

at  $(x_0, y_0)$ , then  $D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

where  $\nabla f = (f_x, f_y)$  (gradient of  $f$ )

pf:  $\Delta z = (f_x(x_0, y_0) + \varepsilon_1) \Delta x + (f_y(x_0, y_0) + \varepsilon_2) \Delta y$

Let  $x(s) = x_0 + s u_1$ ,  $y(s) = y_0 + s u_2$

$$\Delta x = x(s) - x(0) = s u_1 \quad \Delta s = s - 0$$

$$\Delta y = y(s) - y(0) = s u_2$$

$$\Rightarrow D_{\vec{u}} f(x_0, y_0) = \lim_{\Delta s \rightarrow 0} \frac{\Delta z}{\Delta s}$$

$$= \lim_{\Delta s \rightarrow 0} \left( f_x(x_0, y_0) + 0 \right) \frac{\Delta x}{\Delta s} + \left( f_y(x_0, y_0) + 0 \right) \frac{\Delta y}{\Delta s}$$

$$= f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2 = \nabla f(x_0, y_0) \cdot \vec{u}$$

$$\text{Ex 2 } f(x, y) = x^2 + xy$$

Note:  $f$  is differentiable

$$\text{let } \vec{u} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$D_{\vec{u}} f(1, 2) = ?$$

$$\underline{\text{Sol}} = \nabla f(1, 2) \cdot \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$= \frac{f_x(1, 2)}{\sqrt{2}} + \frac{f_y(1, 2)}{\sqrt{2}}$$

$$f_x = 2x + y, \quad f_y = x$$

$$\text{Ans} = \frac{4+1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

$$\text{Ex 3 } f(x,y) = \begin{cases} \frac{4xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Evaluate  $D_{\vec{u}} f(0,0)$  and  $\nabla f(0,0) \cdot \vec{u}$

Ans:  $f_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x-0} = 0$

$$f_y(0,0) = \lim_{y \rightarrow 0} \frac{f(0,y) - f(0,0)}{y-0} = 0$$

$$D_{\vec{u}} f(0,0) = \lim_{s \rightarrow 0} \frac{f(su_1, su_2) - f(0,0)}{s-0} \quad (\underline{\underline{u_1^2 + u_2^2 = 1}})$$

$$= \lim_{s \rightarrow 0} \frac{4s^3 u_1 u_2}{s^2} = 4u_1 u_2 \neq \nabla f(0,0) \cdot \vec{u}$$

$\underline{\underline{=}}$   
 $(0,0)$

$$\text{Ex 4 } f(x, y) = ax + by + c$$

$$\nabla f(0, 0) \cdot \vec{u} \neq D_{\vec{u}} f(0, 0)$$

$$\text{Sol } f_x(0, 0) = a, \quad f_y(0, 0) = b$$

$$D_{\vec{u}} f(0, 0) = \lim_{s \rightarrow 0} \frac{(as u_1 + bs u_2 + c) - c}{s}$$

$$= a u_1 + b u_2 = \nabla f(0, 0) \cdot \vec{u}$$

Remark.  $f$  is diff. at  $(x_0, y_0)$

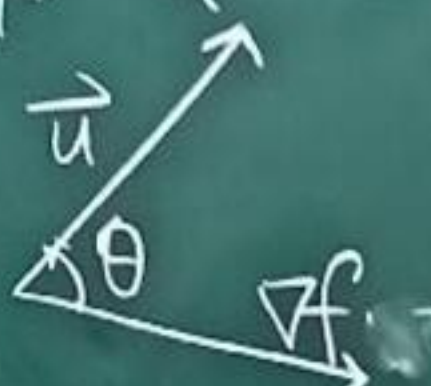
$\Rightarrow z = f(x, y)$  has a tangent plane at  $(x_0, y_0, f(x_0, y_0))$

$\Rightarrow f$  is close to a linear function

$$\Rightarrow D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

# Properties of $\nabla f$

If  $f$  is diff. at  $(x_0, y_0)$

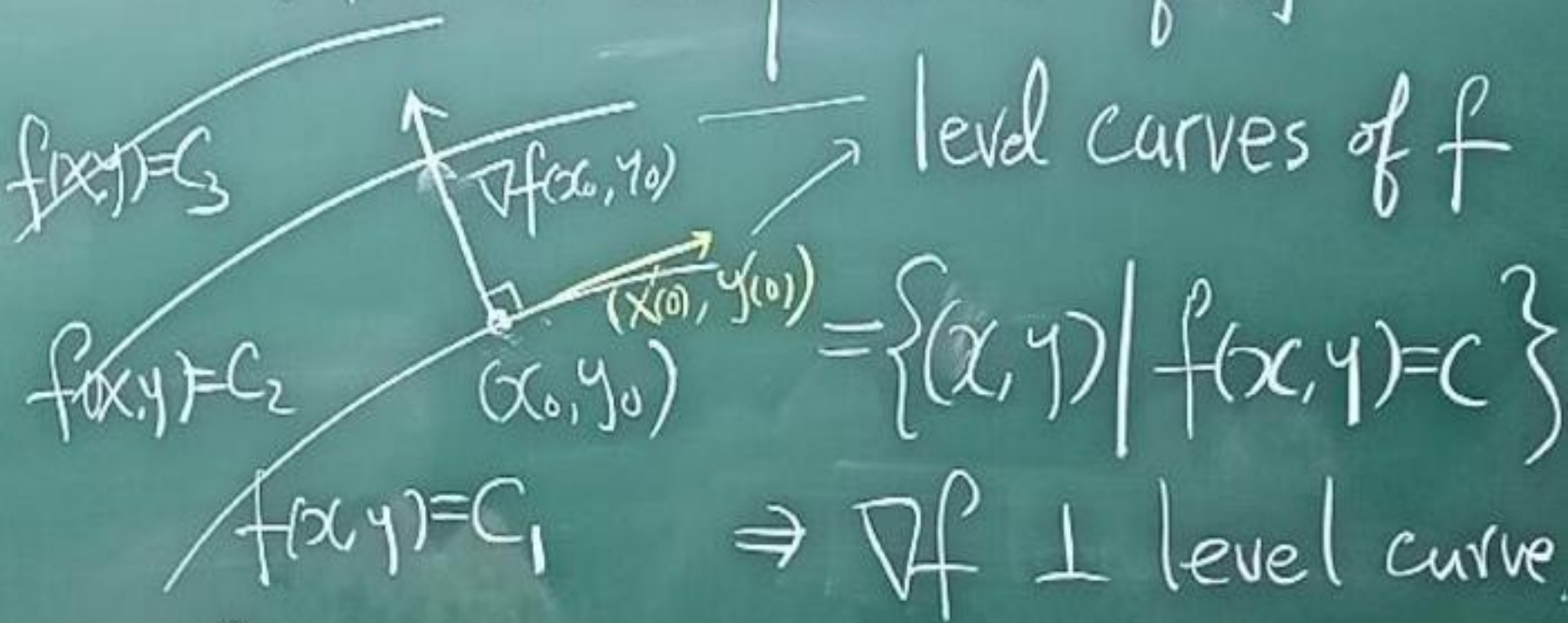
$$\begin{aligned}\Rightarrow D_{\vec{u}} f(x_0, y_0) &= \nabla f(x_0, y_0) \cdot \vec{u} \\ &= |\nabla f(x_0, y_0)| \cdot |\vec{u}| \cos \theta\end{aligned}$$


The diagram shows two vectors originating from a common point. One vector is labeled  $\vec{u}$  and the other is labeled  $\nabla f$ . The angle between them is labeled  $\theta$ .

$\Rightarrow$  (1)  $f$  <sup>increase</sup> most rapidly  
(decrease) in the direction of  $\vec{u}$  if  $\cos \theta = 1$   
(-1)  
i.e. in the direction of  $\nabla f$   
( $-\nabla f$ )

$$(2) D_{\vec{u}} f(x_0, y_0) = 0 \iff \nabla f(x_0, y_0) \perp \vec{u}$$

(3) Another interpretation of  $\nabla f$



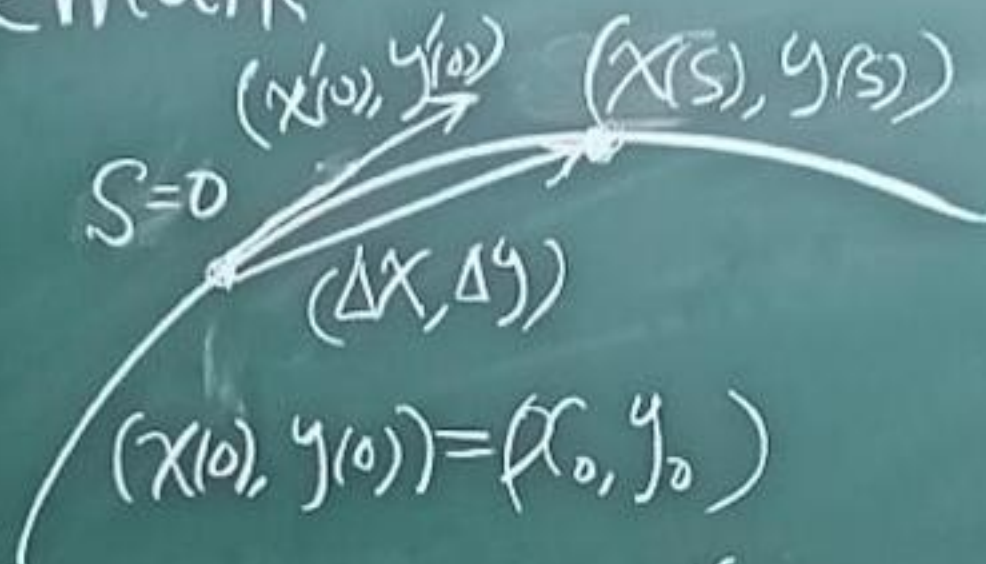
pf: Let  $(x(s), y(s))$  be a level curve of  $f$  (differentiable) with  $(x(0), y(0)) = (x_0, y_0)$

$$\Rightarrow f(x(s), y(s)) = \text{constant}$$

$$\frac{d}{ds} \Big|_{s=0} \Rightarrow \nabla f(x_0, y_0) \cdot (x'(0), y'(0)) = 0$$

$(x'(0), y'(0)) =$  a tangent vector of level curve

Remark



$$\begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix} = \lim_{s \rightarrow 0} \begin{pmatrix} \frac{x(s) - x(0)}{s - 0} \\ \frac{y(s) - y(0)}{s - 0} \end{pmatrix}$$

$$= \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \quad \Delta s = s - 0$$

= a tangent vector

$\therefore (x'(0), y'(0))$  points in the direction of tangent line at  $(x(0), y(0))$

Eg1 Find the tangent line  
(normal)

of  $\frac{x^2}{4} + y^2 = 2$  at  $(-2, 1)$

Sol Let  $f(x, y) = \frac{x^2}{4} + y^2$

$\therefore \frac{x^2}{4} + y^2 = 2$  is a level curve  
of  $f$

$\therefore \nabla f(-2, 1) =$  normal vector of  
the level curve at  $(-2, 1)$

tangent line:  $(-1, 2)$

$$(x+2, y-1) \cdot \nabla f(-2, 1) = 0$$

Normal line:

$$\frac{y-1}{x-2} = \frac{2}{-1}$$

# Algebraic Rule for gradient

$$(i) \nabla(f \pm g) = \nabla f \pm \nabla g$$

$$(ii) \nabla(c f(x, y)) = c \nabla f(x, y)$$

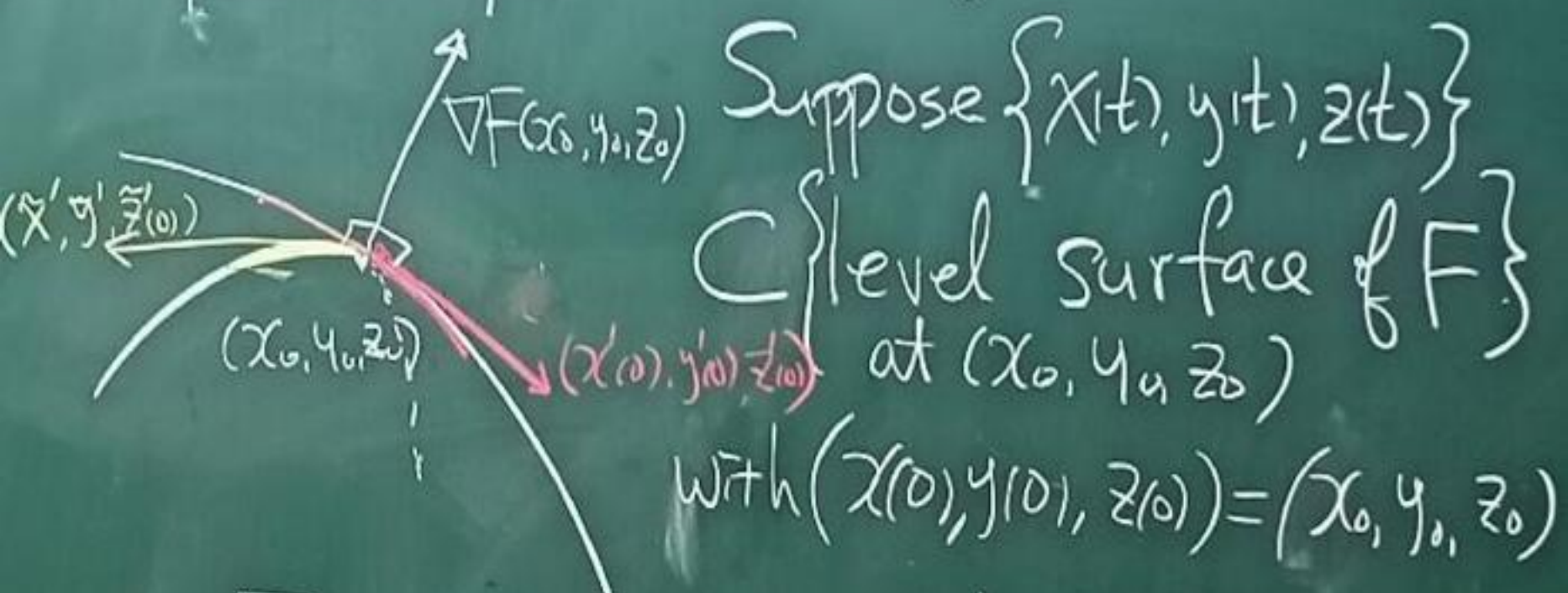
$$(iii) \nabla(f \cdot g) = f \nabla g + g \nabla f$$

$$(iv) \nabla\left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2}$$

**Pf:** Check  $x$ -component  
and  $y$ -component directly

Tangent plane and normal line  
of level surface of  $F(x, y, z)$  (diff.)

$$\{ (x, y, z) \mid F(x, y, z) = C \}$$



$$\Rightarrow F(x(t), y(t), z(t)) = \text{constant} = F(x_0, y_0, z_0)$$

$$\frac{d}{dt} \Big|_{t=0} \Rightarrow \nabla F(x_0, y_0, z_0) \cdot \underbrace{(x'(0), y'(0), z'(0))}_{\text{a tangent vector}}$$

of the level surface at  $(x_0, y_0, z_0)$

For any curve  $(x(t), y(t), z(t))$

on the surface passing  $(x_0, y_0, z_0)$

$$\nabla F(x_0, y_0, z_0) \perp (x'(0), y'(0), z'(0))$$

$$\Rightarrow \nabla F(x_0, y_0, z_0) \perp \begin{array}{l} \text{any tangent} \\ \text{vector at } (x_0, y_0, z_0) \end{array}$$

$$\Rightarrow \nabla F(x_0, y_0, z_0) \perp \begin{array}{l} \text{tangent plane at} \\ (x_0, y_0, z_0) \end{array}$$

tangent plane

$$(x-x_0, y-y_0, z-z_0) \cdot \nabla F(x_0, y_0, z_0) = 0$$

$$\text{normal line: } \frac{x-x_0}{F_x(x_0, y_0, z_0)} = \frac{y-y_0}{F_y(x_0, y_0, z_0)} = \frac{z-z_0}{F_z(x_0, y_0, z_0)}$$

$$\text{or } \begin{cases} x(t) = x_0 + F_x(x_0, y_0, z_0)t \\ y(t) = y_0 + F_y(x_0, y_0, z_0)t \\ z(t) = z_0 + F_z(x_0, y_0, z_0)t \end{cases}$$