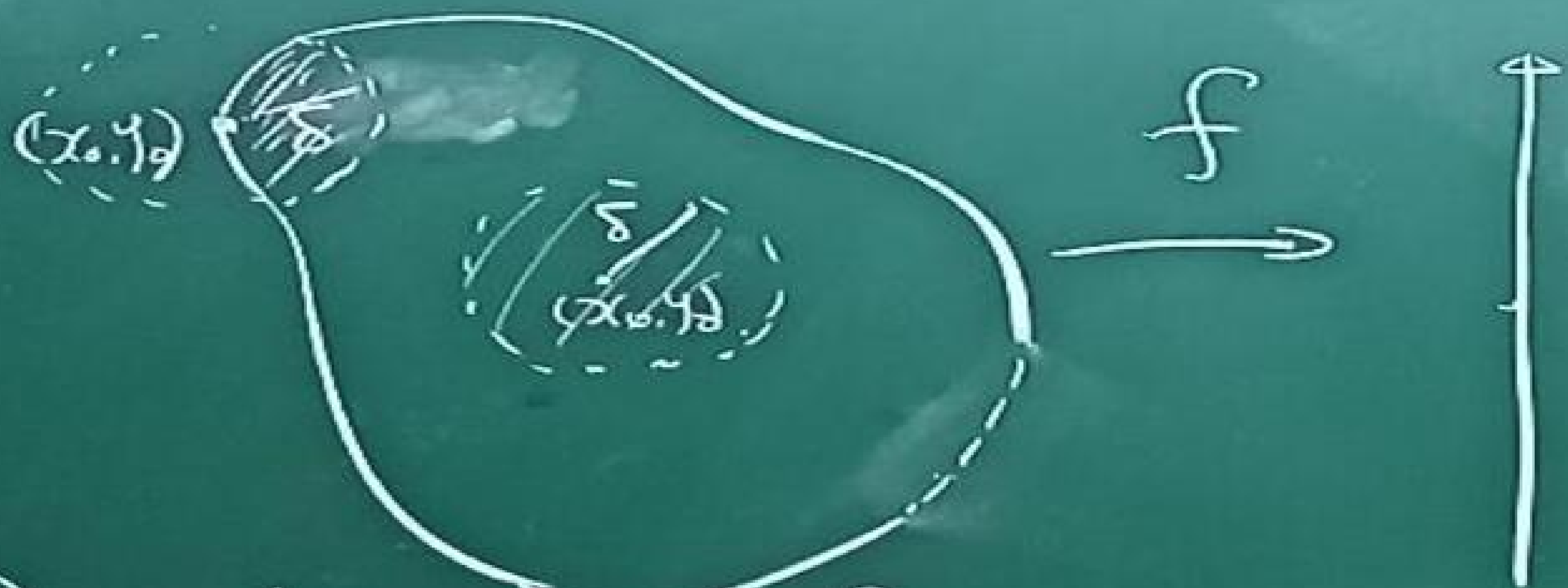
Limit and Continuity in higher dimension

2D case



D_f = domain of f

$$f : D_f \longmapsto \mathbb{R}$$

$$(x, y) \longmapsto z = f(x, y)$$

<u>Def</u> $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$

if for every $\varepsilon > 0$

there exists a corresponding

$\delta > 0$, such that

"$0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Rightarrow |f(x,y)-L| < \varepsilon$"

$+ ((x,y) \in D_f)$ (*)

<u>Remark</u>: If (*) is changed to

"$\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Rightarrow |f(x,y)-L| < \varepsilon$" (**)

then $\begin{cases} \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L \\ f(x_0,y_0) = L \end{cases}$

Def f is cont. at (x_0, y_0)

(ii) lim exists (i) is defined

if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) \underset{\text{(iii)}}{=} f(x_0,y_0)$

That is, for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$, such that

" $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Rightarrow |f(x,y) - f(x_0,y_0)| < \varepsilon$ "

(or $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$)

+ $\boxed{(x,y) \in D_f}$

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L \qquad \text{and} \qquad \lim_{(x, y)\to(x_0, y_0)} g(x, y) = M.$$

1. *Sum Rule:* $$\lim_{(x, y)\to(x_0, y_0)} (f(x, y) + g(x, y)) = L + M$$
2. *Difference Rule:* $$\lim_{(x, y)\to(x_0, y_0)} (f(x, y) - g(x, y)) = L - M$$
3. *Constant Multiple Rule:* $$\lim_{(x, y)\to(x_0, y_0)} kf(x, y) = kL \qquad \text{(any number } k\text{)}$$
4. *Product Rule:* $$\lim_{(x, y)\to(x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$$
5. *Quotient Rule:* $$\lim_{(x, y)\to(x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}, \qquad M \neq 0$$
6. *Power Rule:* $$\lim_{(x, y)\to(x_0, y_0)} [f(x, y)]^n = L^n, \; n \text{ a positive integer}$$
7. *Root Rule:* $$\lim_{(x, y)\to(x_0, y_0)} \sqrt[n]{f(x, y)} = \sqrt[n]{L} = L^{1/n},$$
   n a positive integer, and if n is even, we assume that $L > 0$.

Eg 1 $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x + y^2} = ?$

$\underline{\text{Ans}} = \dfrac{0 - 0\cdot 1 + 3}{0 + 1^2} = 3$

Here we have used the facts that, x, xy, y^2 are continuous.

Eg 2 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = ?$

$\displaystyle = \lim \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$

$\displaystyle = \lim x(\sqrt{x} + \sqrt{y}) = 0$

$\underline{\text{Rm}}$. $D_f = \{x \geq 0,\ y \geq 0,\ x \neq y\}$

Eg 3. $\lim\limits_{\substack{(x,y)\to(0,0) \\ r\to 0}} \dfrac{4xy^2}{x^2+y^2} = ?$

"$0<r<\delta \Rightarrow |f(x,y)-?|<\varepsilon$"

<u>Ans:</u> $x = r\cos\theta$, $y = r\sin\theta$.

$$= \lim_{r\to 0} \frac{4r^3\cos\theta\sin^2\theta}{r^2} = 0$$

<u>Rm</u>

"$0<\sqrt{x^2+y^2}<\delta \Rightarrow |f(x,y)-L|<\varepsilon$"

$\Longleftrightarrow$

"$0<r<\delta \Rightarrow |f(r\cos\theta, r\sin\theta)-L|<\varepsilon$"

i.e. $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{r\to 0} f(r\cos\theta, r\sin\theta)$ (indep of θ)

Eg4 $\lim\limits_{(x,y)\to(0,0)} \frac{y}{x} = ?$

Ans: $x = r\cos\theta,\ y = r\sin\theta$

$$\lim_{r\to 0} \frac{r\sin\theta}{r\cos\theta} \stackrel{?}{=} \underset{(\text{indep of } \theta)}{L}$$

$$= \lim_{r\to 0} \frac{\sin\theta}{\cos\theta} \text{ does not exist.}$$

or Two Path Theorem (See below)

$$\lim_{(x,y)\to(0,0),\, x=y} \frac{y}{x} = 1$$

$$\lim_{(x,y)\to(0,0),\, x=-y} \frac{y}{x} = -1$$

$\Longrightarrow \lim\limits_{(x,y)\to(0,0)}$ does not exist

Two Path Theorem

If $f(x,y)$ have different limits (one dimensional limits) along two Paths passing through (x_0, y_0) as $(x,y) \to (x_0, y_0)$

Then $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist

Eg 5: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$

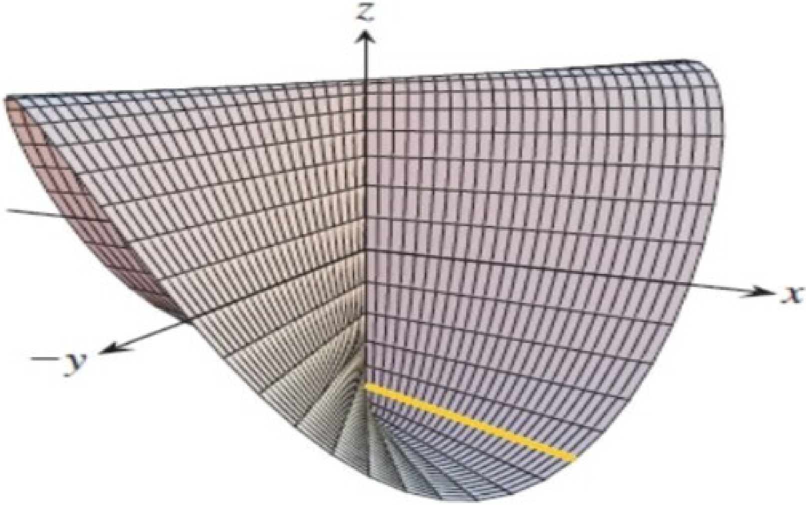
See Fig 14.1.4 page 819

Sol. Let $(x,y)\to(0,0)$ along $y=mx$, $m\in\mathbb{R}$ $\underline{(*)}$

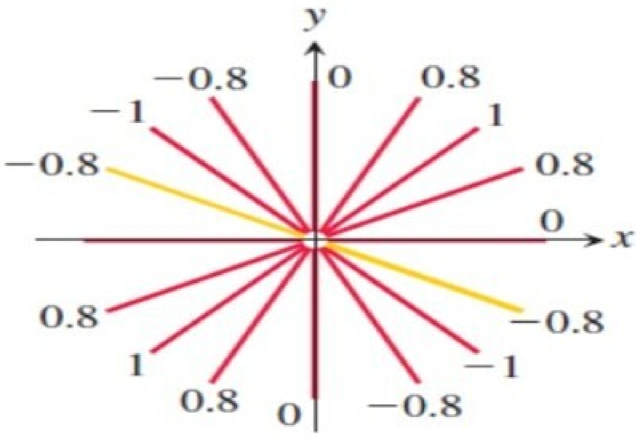
$$\lim_{(*)} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

different $m\in\mathbb{R}$ give different one-dimensional limit.

Two Path Thm $\Rightarrow$ $\displaystyle\lim_{(x,y)\to(0,0)}$ does not exist



(a)



Eg 6. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2} = ?$

See Fig 14.15 page 820

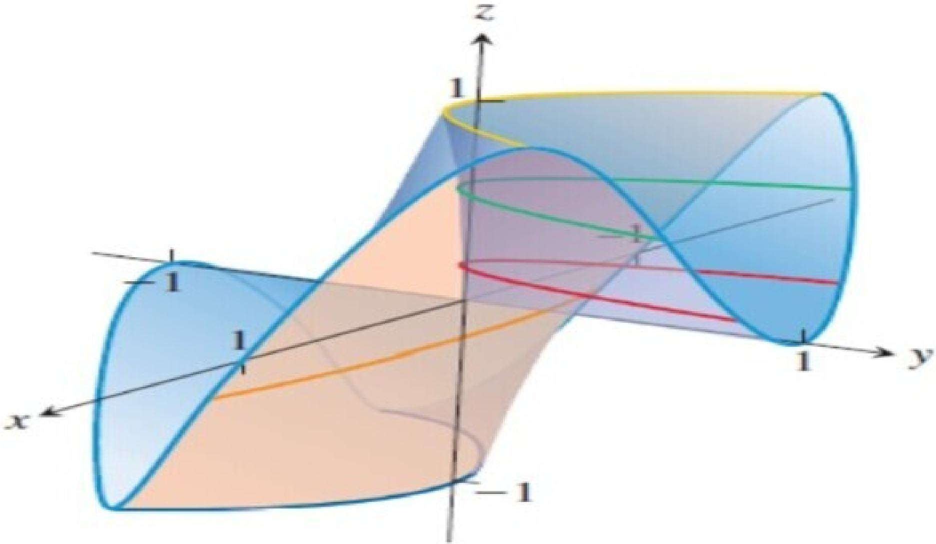
Sol. Let $y = mx$, $x \to 0$

$$\lim_{x\to 0} \frac{2x^2 mx}{x^4+m^2x^2} = \begin{cases} 0, & m=0 \quad (\text{line "} y=0 \text{"}) \\ 0, & m \neq 0 \\ 0, & m=\infty \quad (\text{i.e. line "} x=0 \text{"}) \end{cases}$$

However, if we let $y = kx^2$, $x \to 0$

$$\lim_{x\to 0} \frac{2kx^4}{x^4(1+k^2)} = \frac{2k}{1+k^2} \quad \left(\begin{array}{l} \text{different } k \\ \text{different lim} \end{array}\right)$$

$\therefore$ Two Path Thm $\Rightarrow \lim\limits_{(x,y)\to(0,0)}$ does not exist



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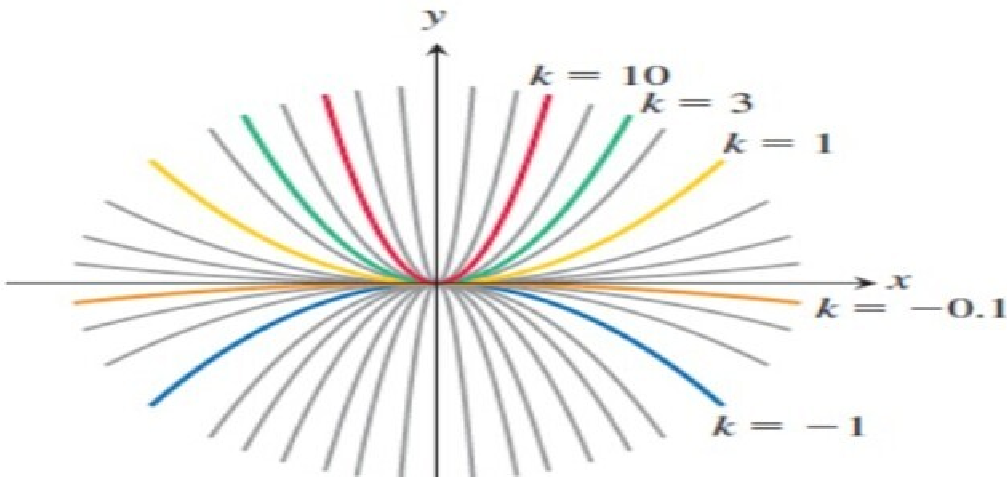


Remark If

$$\begin{cases} f \text{ is cont. at } (x_0, y_0) \\ g \text{ is cont. at } z_0 = f(x_0, y_0) \end{cases}$$

Then $g \circ f$ is cont. at (x_0, y_0)

$$\left(g \circ f(x,y) \overset{\text{def}}{=\!=} g(f(x,y)) \right)$$

$$\begin{tikzcd} \begin{matrix} (x,y) \\ \cdot \\ (x_0,y_0) \end{matrix} \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \begin{matrix} z = f(x,y) \\ \longrightarrow z \\ f(x_0,y_0) \end{matrix} \arrow[r, "g"] & \begin{matrix} w = g(z) \\ \longrightarrow w \\ g(f(x_0,y_0)) \end{matrix} \end{tikzcd}$$

Eg7 $\cos\left(\frac{xy}{x^2+1}\right)$ is continuous on $\mathbb{R}^2$

$$(x,y) \xrightarrow{f} z = \frac{xy}{x^2+1} \xrightarrow{g} W = \cos z$$

$$\lim_{(x,y) \to (x_0,y_0)} \cos\left(\frac{xy}{x^2+1}\right) = \cos\left(\frac{x_0 y_0}{x_0^2+1}\right)$$

Partial derivatives

Def: $\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$

$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$

Notation: $\frac{\partial f}{\partial x} = f_x = \partial_x f = \partial_1 f$

$\frac{\partial f}{\partial y} = f_y = \partial_y f = \partial_2 f$

Eg 1: $f(x, y) = x^2 + 3xy + y - 1$

$\partial_x f = 2x + 3y + 0$

$\partial_y f = 0 + 3x + 1$

Eg2 $f(x,y,z) = x\sin(y+3z)$

$$\partial_x f \underset{y,z\sim\text{constant}}{=\!=\!=} \sin(y+3z)$$

$$\partial_y f \underset{x,z\sim\text{const}}{=\!=\!=} x\cos(y+3z)$$

$$\partial_z f \underset{x,y\sim\text{const}}{=\!=\!=} x\underbrace{\cos(y+3z)}_{\sin'}\cdot \underset{\partial_z(y+3z)}{\overset{3}{\shortparallel}}$$

Eg3 Find $\frac{\partial z}{\partial x}$ at $(0,1,1)$

if $z(x,y)$ is implicitly

defined by $yz + \ln z = x + y$

<u>Ans</u>: $y \cdot z(x,y) + \ln z(x,y) = x + y$

$$\partial_x \Rightarrow y \cdot z_x + \frac{z_x}{z} = 1 + 0$$

$$\Rightarrow \frac{\partial z}{\partial x}(x,y) = \frac{1}{y + \frac{1}{z}}$$

Check $(0,1,1)$ is on this surface

(Yes: $1 \cdot 1 + \ln 1 = 0 + 1$)

$$\therefore \left.\frac{\partial z}{\partial x}\right|_{(0,1,1)} = \frac{1}{1 + \frac{1}{1}} = \frac{1}{2}$$

Higher order Partial derivatives

$$z = f(x,y)$$

$$\frac{\partial^2 f}{\partial x^2} = \partial_x^2 f = \partial_x(\partial_x f) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \partial_y^2 f = \partial_y(\partial_y f) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \partial_x(\partial_y f) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \partial_y(\partial_x f) = f_{xy}$$

Similarly $\partial_y \partial_x^2 f = f_{xxy}$, etc.

Eg4 : $f(x,y) = x^2 + y^2$

$$\partial_x f = 2x, \quad \partial_y f = 2y$$

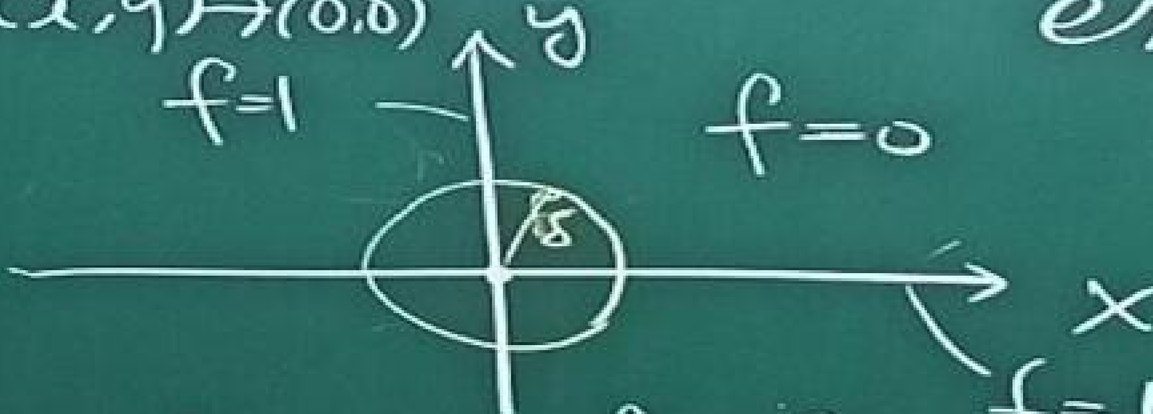
$$\partial_x^2 f = \partial_x(2x) = 2, \quad \partial_y^2 f = 2$$

$$\partial_x \partial_y f = \partial_x(2y) = 0, \quad \partial_y(\partial_x f) = \partial_y(2x) = 0$$

Eg 5. $\left.\begin{array}{l}\partial_x f(x_0, y_0) \\ \partial_y f(x_0, y_0)\end{array}\right\}$ exist $\not\Rightarrow$ f is cont. at (x_0, y_0)

$$f(x,y) = \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$

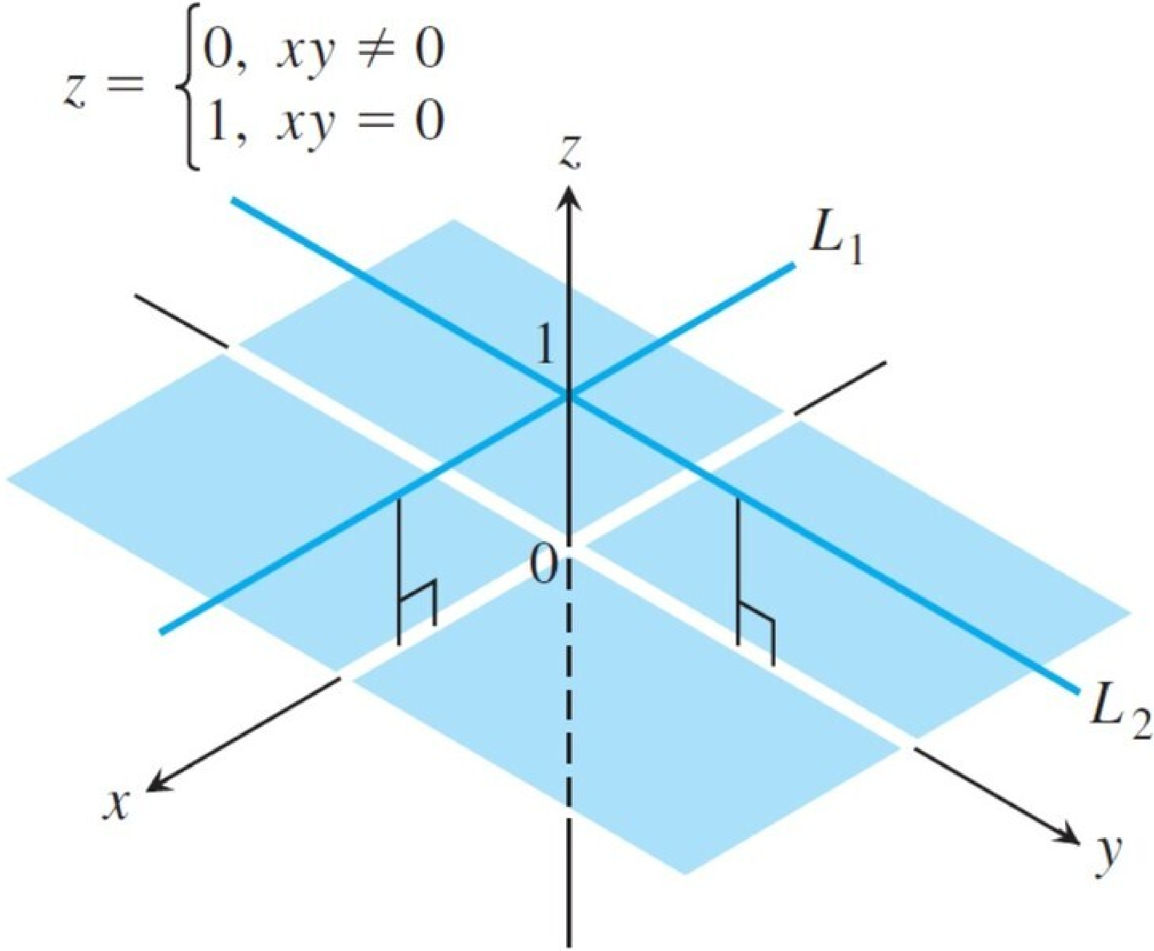
(i) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.



(ii) $\partial_x f(0,0) = \lim\limits_{x\to 0} \dfrac{f(x,0)-f(0,0)}{x-0}$

$= \lim\limits_{x\to 0} \dfrac{1-1}{x-0} = 0$

(iii) $\partial_y f(0,0) = \lim\limits_{y\to 0} \dfrac{f(0,y)-f(0,0)}{y-0} = 0$



$z = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$
z
L_1
1
0
L_2
x
y