Pf of term by term integration :

$$\int_a^x \lim_{N\to\infty} \sum_{n=0}^{N} a_n(t-a)^n\,dt \overset{?}{=} \underbrace{\lim_{N\to\infty} \sum_{n=0}^{N}}_{\sum_{n=0}^{\infty}} \int_a^x a_n(t-a)^n\,dt$$

$$|x-a|<R \qquad \sum_{n=0}^{N}\int_a^x = \int_a^x \sum_{n=0}^{N}$$

$$\int_a^x S(t)\,dt \overset{?}{=} \lim_{N\to\infty} \int_a^x S_N(t)\,dt$$

$$\lim_{N\to\infty} \int_a^x (S(t)-S_N(t))\,dt \overset{?}{=} 0 \quad (*)$$

Note. $|t-a|<|x-a|<R \Rightarrow \lim_{t\to\infty} S_N(t) = S(t)$

(**) $\begin{cases} \text{Given } \varepsilon>0, \exists \text{ corresponding} \\ M>0 \text{ (M may depend on } t\text{)} \\ \text{such that} \\ N>M \Longrightarrow |S_N(t)-S(t)|<\varepsilon \end{cases}$

Def: $S_N(t)$ converges to $S(t)$ uniformly for t in a set D if M in (**) can be chosen independent of $t \in D$

Eg: $f_N(x) = \frac{1}{Nx}$, $x \in (0,1)$

$$\lim_{N\to\infty} f_N(x) = 0$$

But not uniformly on $(0,1)$

Since "$\max_{x\in(0,1)}$" $|f_N(x) - 0| = \infty$ for any $N>0$

Also $\lim_{N\to\infty} |f_N(x) - 0| = \lim_{N\to\infty} \infty \neq 0$

Lemma 1 If $\lim_{N\to\infty} a_n(t-a)^N = S(t)$ converges (absoutely) on $|t-a|<R$

Then it converges uniformly on $|t-a|\le r$ for any $0<r<R$

pf Let $y=a+r'$, $r<r'<R$

$$|t-a|\le r<r'<R$$

$$\lim_{N\to\infty} S_N(y) = S(y)$$

$$\longrightarrow \lim_{N\to\infty} |a_N(y-a)^N| = 0$$

$$\therefore |a_N(y-a)^N| < 1 \text{ if } N>M$$

$$\Rightarrow |a_N(t-a)^N| = |a_N(y-a)^N| \left|\left(\frac{t-a}{y-a}\right)^N\right|$$

$\therefore \left| a_N (t-a)^N \right| \le \left(\frac{|t-a|}{|y-a|} \right)^N$ if $N > M$

$$\frac{|t-a|}{|y-a|} = \frac{|t-a|}{r'} \le \left(\frac{r}{r'}\right) \stackrel{\text{def}}{=} q < 1$$

$$\sum \left(\frac{|t-a|}{|y-a|} \right)^n = \text{convergent Geometric Series}$$

$$\left| S_N(t) - S(t) \right| \le \sum_{M+1}^{\infty} q^M = \frac{q^{M+1}}{1-q} \text{ if } N > M$$

$\forall\, \varepsilon > 0, \exists M$ s.t. $\frac{q^{M+1}}{1-q} < \varepsilon$

$\Longrightarrow$ "$N > M \Rightarrow \left| S_N(t) - S(t) \right| < \varepsilon$" #

Conclusion: $S_N(t)$ converges uniformly on $|t-a| \le |x-a|$ to $S(t)$

$\therefore \left| \int_a^x S_N(t) - S(t)\, dt \right| \le |x-a| \cdot \varepsilon$, $N > M$

$\therefore$ (*) holds