pf of term by term integration

$$\int_a^x \lim_{N\to\infty}\sum_{n=0}^{N} a_n(t-a)^n\,dt \overset{?}{=} \underbrace{\lim_{N\to\infty}\sum_{n=0}^{N}}_{\sum_{n=0}^{\infty}} \int_a^x a_n(t-a)^n\,dt$$

$|x-a|<R$

$$\sum_{n=0}^{N}\int_a^x = \int_a^x \sum_{n=0}^{N}$$

$$\int_a^x S(t)\,dt \overset{?}{=} \lim_{N\to\infty}\int_a^x S_N(t)\,dt$$

$$\lim_{N\to\infty}\int_a^x \left(S(t)-S_N(t)\right)dt \overset{?}{=} 0 \quad (*)$$

Note. $|t-a|<|x-a|<R \Rightarrow \lim_{t\to\infty} S_N(t)=S(t)$

(**) Given $\varepsilon>0$, $\exists$ corresponding $M>0$ (M may depend on t) such that

$N>M \Longrightarrow |S_N(t)-S(t)|<\varepsilon$

Def: $S_N(t)$ converges to $S(t)$ uniformly for t in a set D if M in (**) can be chosen independent of $t \in D$

Eg: $f_N(x) = \frac{1}{Nx}$, $x \in (0,1)$

$$\lim_{N \to \infty} f_N(x) = 0$$

But not uniformly on $(0,1)$

Since "$\max_{x \in (0,1)}$" $|f_N(x) - 0| = \infty$ for any $N > 0$

Also $\lim_{N \to \infty} |f_N(x) - 0| = \lim_{N \to \infty} \infty \neq 0$

Lemma 1 If $\lim_{N\to\infty} a_n(t-a)^N = S(t)$

converges (absolutely) on $|t-a|<R$

Then it converges uniformly

on $|t-a|\le r$ for any $0<r<R$

pf Let $y=a+r'$, $r<r'<R$

$$|t-a|\le r<r'<R$$

$$\lim_{N\to\infty} S_N(y)=S(y)$$

$$\Rightarrow \lim_{N\to\infty}|a_N(y-a)^N|=0$$

$$|a_N(t-a)^N|<1 \text{ if } N>M$$

$$\Rightarrow |a_N(t-a)^N| = |a_N(y-a)^N|\left|\left(\frac{t-a}{y-a}\right)^N\right|$$

$\because \left| a_N (t-a)^N \right| \le \left(\frac{|t-a|}{|y-a|} \right)^N$ if $N > M$

$$\frac{|t-a|}{|y-a|} = \frac{|t-a|}{r'} \le \left(\frac{r}{r'} \right) \stackrel{\text{def}}{=} q < 1$$

$\sum \left(\frac{|t-a|}{|y-a|} \right)^n$ = convergent Geometric Series

$$|S_N(t) - S(t)| \le \sum_{M+1}^{\infty} q^M = \frac{q^{M+1}}{1-q} \text{ if } N > M$$

$\forall\, \varepsilon > 0, \exists M$ s.t. $\frac{q^{M+1}}{1-q} < \varepsilon$

$\Rightarrow$ "$N > M \Rightarrow |S_N(t) - S(t)| < \varepsilon$" #

Conclusion: $S_N(t)$ converges uniformly on $|t-a| \le |x-a|$ to $S(t)$

$\therefore \left| \int_a^x S_N(t) - S(t)\, dt \right| \le |x-a| \cdot \varepsilon$, $N > M$

$\therefore$ (*) holds