

Binomial Series

$$f(x) = (1+x)^m, m \in \mathbb{R}$$

$$T_{f,0}(x) = ?, \neq f(x)$$

(1) If $m \in \mathbb{N}$, $f(x) = \text{polynomial}$

$$\Rightarrow T_{f,0}(x) = f(x)$$

(2) If $m \notin \mathbb{N}$

$$f^{(k)}(x) = m(m-1)\cdots(m-k+1)(1+x)^{m-k}$$

$$f^{(k)}(0) = m(m-1)\cdots(m-k+1)$$

$$\Rightarrow f(x) = P_n(x) + R_n(x)$$

where $P_n(x) = \sum_{k=0}^n \binom{m}{k} x^k$

$$R_n(x) = \binom{m}{n+1} (1 + C_{n+1}) x^{n+1}$$

$$\therefore T_{f,0}(x) = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

where $\binom{m}{k} = \frac{m(m-1) \dots (m-k+1)}{k!}$

(i) Does $T_{f,0}(x)$ converge?

Ratio Test $\Rightarrow \rho = |x|$
(m fixed, $k \rightarrow \infty$)

$\therefore T_{f,0}(x)$ converges on $|x| < 1$
diverges on $|x| > 1$

(ii) $T_{f,0}(x) \stackrel{?}{=} f(x)$ on $|x| < 1$?

$$\lim R_n(x) = 0?$$

Ans: Not clear if $-1 < x < 0$

It can be shown indirectly
(and not easily) that

$$\lim_{n \rightarrow \infty} R_n(x) = 0 \quad \text{on } |x| < 1$$

(Section 10.10, problem 58)

$$\therefore \underline{(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k}$$

for all $m \in \mathbb{R}$, $|x| < 1$

Important! (A!)

Known Taylor Series ($\frac{x^k}{k!}$)

$$* \frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \dots, \quad |x| < 1$$

$$* e^{\pm x} = 1 \pm x + \frac{x^2}{2!} \pm \frac{x^3}{3!} + \dots, \quad x \in \mathbb{R}$$

$$* \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots, \quad x \in \mathbb{R}$$

$$* \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \quad x \in \mathbb{R}$$

$$* \ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \dots \quad \begin{cases} +: -1 < x \leq 1 \\ -: -1 \leq x < 1 \end{cases}$$

$$* \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots, \quad |x| \leq 1$$

$$* (1 \pm x)^m = 1 \pm mx + \frac{m(m-1)}{2!} x^2 \pm \dots, \quad |x| < 1$$

$$* f(x) = \begin{cases} e^{\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}, \quad \begin{matrix} f_0(x) = 0 \neq f(x) \\ x \neq 0 \end{matrix}$$

$$\text{Eg 1 } T_{\sin^{-1}, 0}(x) = ?$$

$$\text{Sol } (\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \quad |x| < 1$$

$$\sin^{-1} x - \sin^{-1} 0 = \int_0^x (\sin t)' dt$$

(Fundamental Thm of Calc.)

$$= \int_0^x (1-t^2)^{-\frac{1}{2}} dt \quad (\text{Binomial, } m = -\frac{1}{2})$$

$$= \int_0^x \left(1 - \frac{1}{2}(-t^2) + \frac{(\frac{1}{2})(\frac{-3}{2})}{2!}(-t^2)^2 + \dots \right) dt$$

$$= x + \frac{x^3}{6} + \frac{x^5}{40} + \dots, \quad |x| < 1$$

$$(ThmA) = T_{\sin^{-1}, 0}(x)$$

$$\text{Ex 2} \quad \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots, \quad |x| < 1$$

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \dots \quad ?$$

$$\text{Sol} \quad \frac{1}{1+t^2} = 1 - t^2 + \dots + (-1)^n t^{2n}$$

$$(\text{not Taylor's Thm}) \quad \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}, \quad \forall t \in \mathbb{R}$$

$$(\because 1+x+\dots+x^n = \frac{1-x^{n+1}}{1-x}, \quad x = -t^2)$$

$$\Rightarrow \tan^{-1} x - \tan^{-1} 0 = \int_0^x \frac{1}{1+t^2} dt$$

$$= \int_0^x \left(1 - t^2 + t^4 - \dots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} \right) dt$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \tilde{R}_n(x)$$

$$\text{Here } \tilde{R}_n(x) = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt$$

$$\text{If } x=1, \quad |\tilde{R}_n(1)| = \int_0^1 \frac{t^{2n+2}}{1+t^2} dt$$

$$\leq \int_0^1 \frac{t^{2n+2}}{1+t^0} dt = \frac{1}{2n+3}$$

(Similarly for $x=-1$)

$$\Rightarrow \lim_{n \rightarrow \infty} \tilde{R}_n(\pm 1) = 0$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$$

Leibnitz formula for π

$$\text{Similarly } \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Remark: If $|t| < 1$

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \dots$$

大小

If $|t| > 1$

$$\frac{1}{1+t^2} = \frac{1}{t^2} \left(\frac{1}{1+\frac{1}{t^2}} \right)$$

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大小

$$= \frac{1}{t^2} \left(1 - \frac{1}{t^2} + \frac{1}{t^4} - \dots \right), |t| > 1$$

Applications:

- (I) Approximation and error estimate
- (II) Alternative method for " $\lim \frac{0}{0}$ "
- (*) (III) Find $f(x)$ from $T_{f,a}(x)$

Eg 3 (Application I)

Find app. value of $\int_0^{\frac{1}{2}} \sin t^2 dt$
and estimate the error.

Ans: $\int_0^{\frac{1}{2}} \sin t^2 dt$

$$= \int_0^{\frac{1}{2}} \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \dots \right) dt$$

$$= \left. \frac{t^3}{3} - \frac{t^7}{7 \cdot 3!} + \frac{t^{11}}{11 \cdot 5!} - \dots \right|_0^{\frac{1}{2}}$$

(Alternating Series)

$$= \frac{1}{3 \cdot 2^3} - \frac{1}{7 \cdot 3! \cdot 2^7} + \frac{1}{11 \cdot 5! \cdot 2^{11}} - \dots$$

$$|E| \leq \frac{\left(\frac{1}{2}\right)^{14}}{15 \cdot 7!}$$

Approximation
(error estimate of Alternating Series)

Eg4 (App II)

$$\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^3} = ?$$

Sol. Method 1. l'Hôpital.

Method 2.

$$f(x) = \frac{\sin x - \frac{\sin x}{\cos x}}{x^3}$$

$$= \frac{\frac{1}{2} \sin 2x - \sin x}{x^3 \cos x}$$

$$= \frac{\frac{1}{2} (2x - \frac{(2x)^3}{3!} + \dots) - (x - \frac{x^3}{3!} + \dots)}{x^3 \cos x}$$

$$= \frac{x^3 (1 - \frac{x^2}{2!} + \dots)}{x^3 (\frac{-4}{3!} + \frac{1}{3!} + \dots)}$$

$$x \rightarrow 0 \Rightarrow \text{Ans} = \frac{-1}{2}$$

$$\text{Eg 5 (App II)} \quad \lim_{x \rightarrow 0} \frac{1}{x} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$$

$$= \lim_{x \rightarrow 0} \frac{x - \sin x}{x^2 \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)}{x^2 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)}$$

$$= \lim_{x \rightarrow 0} \frac{x^3 \left(\frac{1}{3!} + x^2 \left(\frac{1}{5!} + a_2 x^2 + \dots \right) \right)}{x^3 \left(1 + x^2 \left(\frac{1}{3!} + b_2 x^2 + \dots \right) \right)}$$

Both $A(x) = \frac{1}{5!} + a_2 x^2 + a_4 x^4 + \dots$

$B(x) = \frac{1}{3!} + b_2 x^2 + b_4 x^4 + \dots$

are convergent power series

$$= \frac{\frac{1}{3!} + 0 \cdot A(0)}{1 + 0 \cdot B(0)} = \frac{1}{3!}$$

Eg 6 (App III, find $f(x)$ from $\sum a_n(x-a)^n$)

$$\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \dots = ?$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \left| x = \frac{1}{2} \right.$$

$$= -\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n} \quad \left| x = \frac{1}{2} \right.$$

$$= \int_0^x \sum_{n=1}^{\infty} (-t)^{n-1} dt \quad \left| x = \frac{1}{2} \right.$$

$$= \int_0^x (-t + t^2 - t^3 + \dots) dt \quad \left| x = \frac{1}{2} \right.$$

$$= \int_0^{\frac{1}{2}} \frac{1}{1+t} dt = \ln(1+t) \Big|_0^{\frac{1}{2}} = \ln\left(\frac{3}{2}\right)$$

Method 2:

$$\text{or Let } f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (*)$$

$$\Rightarrow f'(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \dots = \frac{1}{1+x}$$

($|x| < 1$, ratio or root test)

$$\Rightarrow f(x) = \ln(1+x) + C$$

$$(*)_{x=0} \Rightarrow f(0) = 0 = C$$

$$\therefore f(x) = \ln(1+x)$$

$$\text{We want } f\left(\frac{1}{2}\right) = \ln\left(\frac{3}{2}\right)$$

A

Eg 9, Application (II):

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2} x^n, \quad R = ?$$

(Follow up on Exercise 10.7, problem 40)

Ans: $R = e$

ie. converges if $|x| < e$

Conv/div at $x = \pm e$?

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2} e^n \quad (x=e)$$

||

$$\sum_{n=1}^{\infty} \left(\frac{e}{\left(\frac{n+1}{n} \right)^n} \right)$$

$\sum_{n=1}^{\infty} a_n$ a_n

$$a_n = \left(\frac{e}{\left(\frac{n+1}{n}\right)^n} \right)^n$$

$$= \left(e \cdot \left(1 + \frac{1}{n}\right)^{-n} \right)^n$$

$$= e^{(1 - n \ln(1 + \frac{1}{n})) \cdot n}$$

$$\left(n - n^2 \left(\frac{1}{n} - \frac{1}{2n^2} + \dots \right) \right)$$

$$= e$$

$$\left(\frac{1}{2} - \frac{1}{3n} + \dots \right)$$

$$= e$$

$$\therefore \lim_{n \rightarrow \infty} a_n = e^{\frac{1}{2}} \neq 0 \text{ div}$$

Alternative Proof (l'Hôpital)

$$a_n = e^{(1 - n \ln(1 + \frac{1}{n})) \cdot n}$$

$$\lim_{n \rightarrow \infty} (1 - n \ln(1 + \frac{1}{n})) \cdot n \quad (= (1 - \infty \cdot 0) \cdot \infty)$$

$$= \lim_{n \rightarrow \infty} (n - n^2 \ln(1 + \frac{1}{n}))$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{1}{n} - \ln(1 + \frac{1}{n})}{\frac{1}{n^2}}$$

$$= \lim_{x \rightarrow 0^+} \frac{x - \ln(1+x)}{x^2}$$

l'Hôpital twice $= \frac{1}{2}$

$$\therefore \lim_{n \rightarrow \infty} a_n = e^{\frac{1}{2}} \neq 0. \text{ div}$$

Eg 7 $\sum_{n=1}^{\infty} n^2 x^n = ?$ (See lecture 08)
 $|x| < 1$

Eg 8 $\sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} = ?$ $|x| > 1$

$= \sum_{n=1}^{\infty} n(n+1) y^n$, $y = \frac{1}{x}$, $|y| < 1$

$f(y) = \sum_{n=1}^{\infty} y^n = \frac{1}{1-y} - 1 = \frac{-y}{1-y}$

$y f'(y) = \sum_{n=1}^{\infty} n y^{n-1} \cdot y$

$y^2 f''(y) = \sum_{n=1}^{\infty} n(n-1) y^{n-2} \cdot y^2$

Ans = $\sum_{n=1}^{\infty} n(n+1) y^n = 2y f'(y) + y^2 f''(y)$

$= \frac{2y}{(1-y)^2} + \frac{2y^2}{(1-y)^3} = \frac{2y}{(1-y)^3}$

$y = \frac{1}{x}$, $= \frac{2x}{(x-1)^3}$ $|x| > 1$

Remark (Euler identity)

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R}$$

$i = \sqrt{-1}$

Since

$$e^x \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

can be identically defined
for $z \in \mathbb{C}$ (complex number)

The radius of convergence = R (~~here~~ $= \infty$)

means the series $\begin{cases} \text{converges on } |z| < R \\ \text{diverges on } |z| > R \end{cases}$

$$\therefore e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Converges for all $z \in \mathbb{C}$

$$\begin{aligned} \therefore e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} \\ &\quad + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \right) \\ &\quad + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right) \end{aligned}$$

$$= \cos\theta + i\sin\theta$$

$$\left(\text{check } \frac{d}{d\theta} e^{i\theta} = i e^{i\theta} = i(\cos\theta + i\sin\theta) \right)$$

All algebraic computations
and diff/integ can be
performed on $e^{i\theta}$ directly

$$\text{Eg } \int e^{ax} \cos bx \, dx \quad a, b \in \mathbb{R}$$

$$= \int e^{ax} \operatorname{Re}(e^{ibx}) \, dx$$

$$= \operatorname{Re} \left(\int e^{ax} e^{ibx} \, dx \right)$$

$$= \operatorname{Re} \left(\int e^{(a+ib)x} \, dx \right)$$

$$= \operatorname{Re} \left(\frac{1}{a+ib} e^{(a+ib)x} + C \right)$$

$$= \operatorname{Re} \left(\frac{(a-ib) e^{(a+ib)x}}{(a+ib)(a-ib)} + C \right) = \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2} + C$$