Binomial Series

$f(x) = (1+x)^m, \ m \in \mathbb{R}$

$T_{f,0}(x) = ? \overset{?}{=} f(x)$

(1) If $m \in \mathbb{N}$, $f(x) =$ polynomial

$\Rightarrow T_{f,0}(x) = f(x)$

(2) If $m \notin \mathbb{N}$

$f^{(k)}(x) = m(m-1)\cdots(m-k+1)(1+x)^{m-k}$

$f^{(k)}(0) = m(m-1)\cdots\cdots(m-k+1)$

$\Rightarrow f(x) = P_n(x) + R_n(x)$

where $P_n(x) = \sum_{k=0}^{n} \binom{m}{k} x^k$

$R_n(x) = \binom{m}{n+1} (1+C_{n+1})^{m-n-1} x^{n+1}$

$\therefore\ T_{f,0}(x) = \sum_{k=0}^{\infty} \binom{m}{k} x^k$

where $\binom{m}{k} = \dfrac{m(m-1)\cdots(m-k+1)}{k!}$

(i) Does $T_{f,0}(x)$ converge?

Ratio Test $\Rightarrow \rho = |x|$

(m fixed, $k \to \infty$)

$\therefore\ T_{f,0}(x)$ converges on $|x|<1$, diverges on $|x|>1$

(ii) $T_{f,0}(x) \overset{?}{=} f(x)$ on $|x|<1$?

$\lim R_n(x) = 0$?

<u>Ans</u>: Not clear if $-1<x<0$

It can be shown indirectly (and not easily) that

$$\lim_{n\to\infty} R_n(x) = 0 \quad \text{on } |x|<1$$

(Section 10.10, problem 58)

$$\therefore \ (1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

for all $m \in \mathbb{R}$, $|x|<1$

Important! (背!)

Known Taylor Series (암기!)

* $\dfrac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \cdots, \quad |x| < 1$
* $e^{\pm x} = 1 \pm x + \dfrac{x^2}{2!} \pm \dfrac{x^3}{3!} + \cdots \quad x \in \mathbb{R}$
* $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots \quad x \in \mathbb{R}$
* $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots \quad x \in \mathbb{R}$
* $\ln(1 \pm x) = \pm x - \dfrac{x^2}{2} \pm \quad \begin{cases} +: -1 < x \le 1 \\ -: -1 \le x < 1 \end{cases}$
* $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots \quad |x| \le 1$
* $(1 \pm x)^m = 1 \pm mx + \dfrac{m(m-1)}{2!} x^2 \pm \dfrac{m(m-1)(m-2)}{3!} x^3 + \cdots \quad |x| < 1$
* $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \ne 0 \\ 0 & x = 0 \end{cases}, \quad T_{f,0}(x) = 0 \ne f(x) \quad x \ne 0$

Eg 1 $T_{\sin^{-1},0}(x) = ?$

Sol $(\sin^{-1}x)' = \dfrac{1}{\sqrt{1-x^2}}$, $|x|<1$

$\sin^{-1}x - \sin^{-1}0 = \int_0^x (\sin^{-1}t)'\,dt$

(Fundamental Thm of Calc)

$= \int_0^x (1-t^2)^{-\frac{1}{2}}\,dt$ (Binomial, $m = -\frac{1}{2}$)

$= \int_0^x \left(1 - \frac{1}{2}(-t^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-t^2)^2 + \cdots\right)dt$

$= x + \dfrac{x^3}{6} + \dfrac{x^5}{40} + \cdots$, $|x|<1$

(Thm A) $= T_{\sin^{-1},0}(x)$

Eg2 $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots,\ |x|<1$

$$\frac{\pi}{4} = \tan^{-1} 1 \overset{?}{=} 1 - \frac{1}{3} + \frac{1}{5} - \cdots ?$$

Sol $\frac{1}{1+t^2} = 1 - t^2 + \cdots + (-1)^n t^{2n}$

(not Taylor's Thm) $+ \frac{(-1)^{n+1} t^{2n+2}}{1+t^2},\ \forall t \in \mathbb{R}$

$$\left(\because 1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x},\ x = -t^2\right)$$

$$\Rightarrow \tan^{-1} x - \tan^{-1} 0 = \int_0^x \frac{1}{1+t^2}\, dt$$

$$= \int_0^x \left(1 - t^2 + t^4 - \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\right) dt$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \tilde{R}_n(x)$$

Here $\tilde{R}_n(x) = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt$

If $x = 1$.

$$|\tilde{R}_n(1)| = \int_0^1 \frac{t^{2n+2}}{1+t^2} dt$$

$$\leq \int_0^1 \frac{t^{2n+2}}{1+0} dt = \frac{1}{2n+3}$$

(Similarly for $x = -1$)

$$\Rightarrow \lim_{n\to\infty} \tilde{R}_n(\pm 1) = 0$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

Leibnitz formula for π

Similarly $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Remark: If $|t|<1$

$$\frac{1}{1+t^2}=1-t^2+t^4\cdots\cdots$$

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If $|t|>1$

$$\frac{1}{1+t^2}=\frac{1}{t^2}\left(\frac{1}{1+\frac{1}{t^2}}\right)$$

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$$=\frac{1}{t^2}\left(1-\frac{1}{t^2}+\frac{1}{t^4}-\cdots\cdots\right),\ |t|>1$$

Applications:

(I) Approximation and error estimate

(II) Altenative method for "$\lim \frac{0}{0}$"

(*)(III) Find $f(x)$ from $T_{f,a}(x)$

Eg3 (Application I)

Find app. value of $\int_0^{\frac{1}{2}} \sin t^2 \, dt$ and estimate the error.

<u>Ans</u>: $\int_0^{\frac{1}{2}} \sin t^2 \, dt$

$= \int_0^{\frac{1}{2}} \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} \cdots \right) dt$

$= \frac{t^3}{3} - \frac{t^7}{7 \cdot 3!} + \frac{t^{11}}{11 \cdot 5!} \cdots \Big|_0^{\frac{1}{2}}$

(Alternating Series

$= \underbrace{\frac{1}{3 \cdot 2^3} - \frac{1}{7 \cdot 3! \cdot 2^7} + \frac{1}{11 \, 5! \, 2^{11}}}_{\text{Approximation}} - \cdots$

$|E| \leq \frac{\left(\frac{1}{2}\right)^{15}}{15 \cdot 7!}$ (error estimate of Alternating Series)

Eg4 (App II)

$$\lim_{x \to 0} \frac{\sin x - \tan x}{x^3} = ?$$

Sol. Method 1: l'Hôpital.

Method 2.

$$f(x) = \frac{\sin x - \frac{\sin x}{\cos x}}{x^3}$$

$$= \frac{\frac{1}{2}\sin 2x - \sin x}{x^3 \cos x}$$

$$= \frac{\frac{1}{2}\left(\cancel{2x} - \frac{(2x)^3}{3!} + \cdots\right) - \left(\cancel{x} - \frac{x^3}{3!} + \cdots\right)}{x^3\left(1 - \frac{x^2}{2!} + \cdots\right)}$$

$$= \frac{\cancel{x^3}\left(\frac{-4}{3!} + \frac{1}{3!} + \cdots\right)}{\cancel{x^3}(1 + \cdots)}$$

$x \to 0 \Rightarrow \text{Ans} = \frac{-1}{2}$

Eg5 (App II) $\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

$$=\lim_{x\to 0}\frac{x-\sin x}{x^2\sin x}$$

$$=\lim_{x\to 0}\frac{\cancel{x}-(\cancel{x}-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots)}{x^2(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots)}$$

$$=\lim_{x\to 0}\frac{\cancel{x^3}\left(\frac{1}{3!}+x^2\left(\frac{-1}{5!}+a_2x^2+\cdots\right)\right)}{\cancel{x^3}\left(1+x^2\left(\frac{-1}{3!}+b_2x^2+\cdots\right)\right)}$$

Both $A(x)=\frac{-1}{5!}+a_2x^2+a_4x^4+\cdots$

$B(x)=\frac{-1}{3!}+b_2x^2+b_4x^4+\cdots\cdots$

are <u>convergent</u> power series

$$=\frac{\frac{1}{3!}+0\cdot A(0)}{1+0\cdot B(0)}=\frac{1}{3!}$$

Eg6 (App III, find $f(x)$ from $\sum a_n(x-a)^n$)

$$\frac{1}{1\cdot 2^1}-\frac{1}{2\,2^2}+\frac{1}{3\,2^3}-\frac{1}{4\cdot 2^4}+\cdots=?$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\Big|_{x=\frac{1}{2}}$$

$$=-\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n}=-\sum_{n=1}^{\infty}\frac{(-x)^n}{n}\Big|_{x=\frac{1}{2}}$$

$$=\int_0^x\sum_{n=1}^{\infty}(-t)^{n-1}\,dt\Big|_{x=\frac{1}{2}}$$

$$=\int_0^x(1-t+t^2-t^3+\cdots)\,dt\Big|_{x=\frac{1}{2}}$$

$$=\int_0^{\frac{1}{2}}\frac{1}{1+t}\,dt=\ln(1+t)\Big|_0^{\frac{1}{2}}=\ln\left(\frac{3}{2}\right)$$

Method 2:

or Let $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (*)$

$\Rightarrow f'(x) = 1 - x + \frac{x^2}{\cancel{2}} - \frac{x^3}{\cancel{3}} + \cdots = \frac{1}{1+x}$

($|x| < 1$, ratio or root test)

$\Rightarrow f(x) = \ln(1+x) + C$

$(*)_{x=0} \Rightarrow f(0) = 0 = C$

$\therefore f(x) = \ln(1+x)$

We want $f\left(\frac{1}{2}\right) = \ln\left(\frac{3}{2}\right)$

Eg 9, Application (II):

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2} x^n, \quad R=?$$

(Follow up on Exercise 10.7, problem 40)

Ans: $R = e$

i.e. converges if $|x| < e$

Conv/div at $x = \pm e$?

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2} e^n \quad (x=e)$$

$$= \sum_{n=1}^{\infty}\left(\frac{e}{\left(\frac{n+1}{n}\right)^n}\right)^n$$

$$\sum_{n=1}^{\infty} a_n \quad a_n = \left(\frac{e}{\left(\frac{n+1}{n}\right)^n} \right)^n$$

$$= \left(e \cdot \left(1+\frac{1}{n}\right)^{-n} \right)^n$$

$$= e^{\left(1 - n\ln\left(1+\frac{1}{n}\right)\right)\cdot n}$$

$$= e^{\left(n - n^2\left(\frac{1}{n} - \frac{1}{2n^2} + \cdots\right)\right)}$$

$$= e^{\left(\frac{1}{2} - \frac{1}{3n} + \cdots\right)}$$

$$\therefore \lim_{n\to\infty} a_n = e^{\frac{1}{2}} \neq 0 \text{ div}$$

Alternative Proof (l'Hôpital)

$$a_n = e^{\left(1 - n\ln\left(1+\frac{1}{n}\right)\right)\cdot n}$$

$$\lim_{n\to\infty}\left(1 - n\ln\left(1+\frac{1}{n}\right)\right)\cdot n \quad \left(= (1-\infty\cdot 0)\cdot\infty\right)$$

$$= \lim_{n\to\infty}\left(n - n^2\ln\left(1+\frac{1}{n}\right)\right)$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)}{\frac{1}{n^2}}$$

$$= \lim_{x\to 0^+}\frac{x - \ln(1+x)}{x^2}$$

l'Hôpital twice $= \frac{1}{2}$

$\therefore \lim_{n\to\infty} a_n = e^{\frac{1}{2}} \neq 0.$ div

(*)

Eg 7 $\sum_{n=1}^{\infty} n^2 x^n = ?$ $\left(\begin{array}{l}\text{See lecture } 08 \\ |x|<1\end{array}\right)$

Eg 8 $\sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} = ?$ $|x|>1$

$= \sum_{n=1}^{\infty} n(n+1) y^n, \quad y = \frac{1}{x}, \quad |y|<1$

$f(y) = \sum_{n=1}^{\infty} y^n = \frac{1}{1-y} - 1 = \frac{-y}{1-y}$

$y f'(y) = \sum_{n=1}^{\infty} n y^{n-1} \cdot y$

$y^2 f''(y) = \sum_{n=1}^{\infty} n(n-1) y^{n-2} \cdot y^2$

$\text{Ans} = \sum_{n=1}^{\infty} n(n+1) y^n = 2y f'(y) + y^2 f''(y)$

$= \frac{2y}{(1-y)^2} + \frac{2y^2}{(1-y)^3} = \frac{2y}{(1-y)^3}$

$y = \frac{1}{x}$ $= 2x/(x-1)^3 \quad |x|>1$

Remark (Euler identity)

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R}$$

$i = \sqrt{-1}$

Since

$$e^{x} \overset{\text{def}}{=\!=} \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

can be identically defined for $z \in \mathbb{C}$ (complex number)

The radius of convergence $= R \left(\begin{smallmatrix} \text{here} \\ = \infty \end{smallmatrix} \right)$

means the series $\begin{cases} \text{converges on } |z| < R \\ \text{diverges on } |z| > R \end{cases}$

$$\because \; e^{z} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Converges for all $z \in \mathbb{C}$

$$\therefore \; e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$$\left(\xRightarrow{\text{check}} \frac{d}{d\theta} e^{i\theta} = i e^{i\theta} = i(\cos\theta + i\sin\theta) \right)$$

All algebraic computations and diff/integ. can be performed on $e^{i\theta}$ directly

Eg $\int e^{ax}\cos bx\,dx \quad a, b \in \mathbb{R}$

$$= \int e^{ax}\,\mathrm{Re}(e^{ibx})\,dx$$

$$= \mathrm{Re}\left(\int e^{ax}e^{ibx}\,dx\right)$$

$$= \mathrm{Re}\left(\int e^{(a+ib)x}\,dx\right)$$

$$= \mathrm{Re}\left(\frac{1}{a+ib}e^{(a+ib)x} + C\right)$$

$$= \mathrm{Re}\left(\frac{(a-ib)e^{(a+ib)x}}{(a+ib)(a-ib)} + C\right) = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C$$