Def. Power Series

$$\sum_{n=0}^{\infty} C_n(x-a)^n \overset{\text{def}}{=\!=} C_0 + \sum_{n=1}^{\infty} C_n(x-a)^n$$

It is a function of x.

a = center (fixed). C_n = coefficients

Question: When "a" and "C_n" are given, for what values of x is the power series convergent?

Eg 1. $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n$. (Geometric Series)

$r = \frac{-(x-2)}{2}$ (div if $|r| \geq 1$)

Power Series converges (abs) $\Leftrightarrow |r| < 1 \Leftrightarrow 0 < x < 4$

Eg2 $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$(0! \stackrel{\text{def}}{=} 1.\ x^0 \stackrel{\text{def}}{=} 1)$

<u>Sol</u>: Ratio test:

$$|u_n| = \frac{|x|^n}{n!}$$

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x|}{n} = 0$$

$\Rightarrow$ For any $x \in \mathbb{R}$

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges (absolutely)

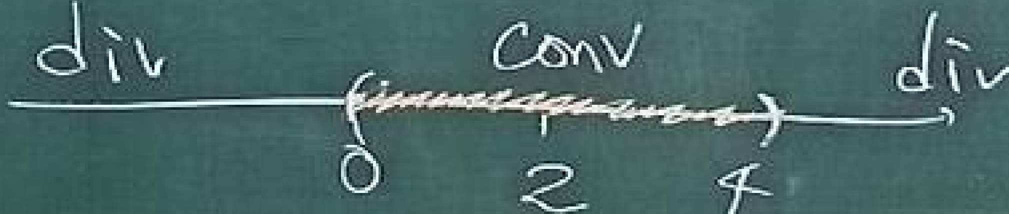
Eg3 $\sum_{n=0}^{\infty} n! x^n$

Ratio Test:

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} n|x| = \begin{cases} 0 & x=0 \\ \infty & x\neq 0 \end{cases}$$

It converges at $x=0$ only and diverges elsewhere.

Eg1



Eg2

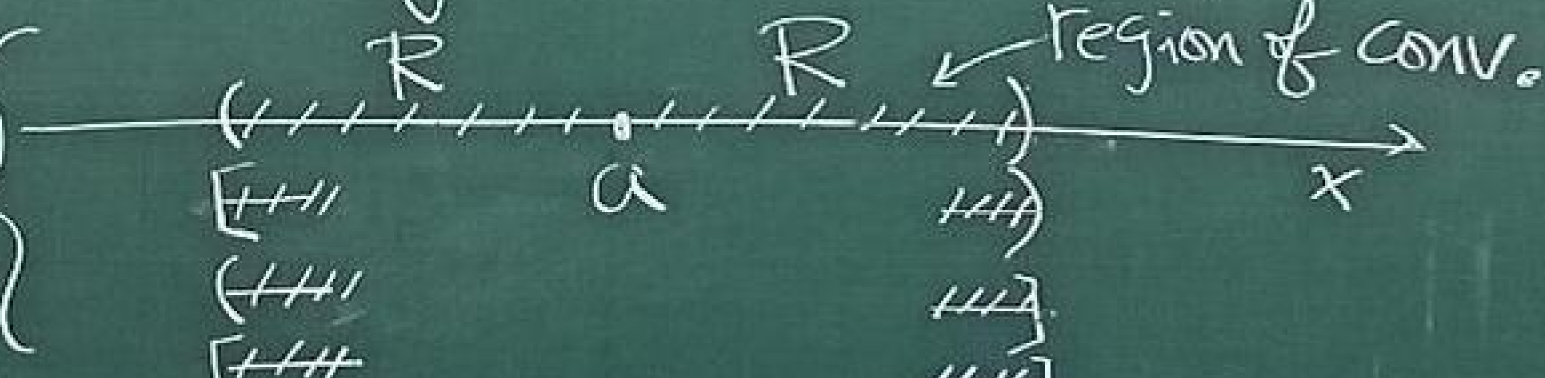
Eg3

We now show that the region of convergence of a Power Series must be an interval centered at the "center of the Power Series". The radius of the interval is called the radius of convergence, usually can be found by Ratio Test or Root Test.

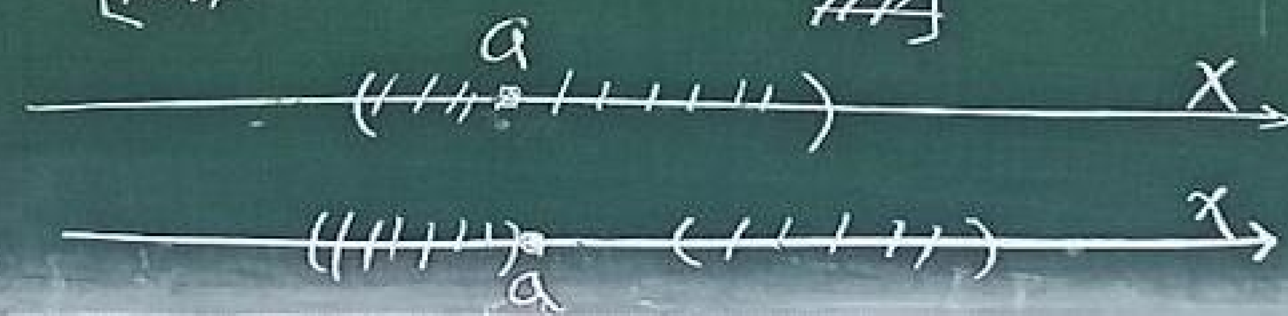
Thm If $\sum_{n=0}^{\infty} a_n x^n$ $\left(\sum a_n(x-a)^n\right)$ converges at $x=c\neq 0$ $(x=c\neq a)$ then it converges absolutely for $|x|<|c|$ $(|x-a|<|c-a|)$

($\Rightarrow$ If it diverges at $x=d$, then it diverges for $|x|>|d|$ $(|x-a|>|d-a|)$)

Possible



Impossible



Pf: If $\sum_{n=0}^{\infty} a_n c^n$ converges

$$\Longrightarrow \lim_{n\to\infty} a_n c^n = 0$$

$$\Rightarrow |a_n c^n| < 1 \quad \forall n \geq N$$

$$\Rightarrow |a_n| < \frac{1}{|c|^n} \quad \forall n \geq N$$

If $|x| < |c|$

$$\Rightarrow \sum_{n=N}^{\infty} |a_n x^n| \leq \sum_{n=N}^{\infty} \left(\frac{|x|}{|c|}\right)^n < \infty$$

$\therefore \sum_{n=0}^{\infty} a_n x^n$ converges absolutely on $|x| < |c|$

In Summary: Possible regions of convergence.

(1) It converges absolutely for all $x \in \mathbb{R}$.
i.e. Radius of conv. $R = \infty$

(2) Convergence only at $x = a$ $(R = 0)$

(3) $\exists\ 0 < R < \infty$

it $\begin{cases} \text{conv. on } |x| < R \\ \text{div. on } |x| > R \end{cases}$

R = radius of conv. for $\sum a_n (x-a)^n$

In general $0 \leq R \leq \infty$

How to find R for $\sum a_n(x-a)^n$?

<u>Ans</u>: If $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = \rho \quad 0 \leq \rho \leq \infty$

$$\Rightarrow \lim_{n\to\infty} \frac{|a_n(x-a)^{n+1}|}{|a_n(x-a)^n|} = \rho\,|x-a|$$

$$\Rightarrow \sum a_n(x-a)^n \begin{cases} \text{conv. abs. if } |x-a| < \frac{1}{\rho} \\ \text{div. if } |x-a| > \frac{1}{\rho} \end{cases}$$

i.e. $R = \dfrac{1}{\rho}$

Similarly. if $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho$

Then $R = \dfrac{1}{\rho}$

Remark: R (radius of conv.) always exists for $\sum_{n=0}^{\infty} a_n(x-a)^n$

but $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ or $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ may or may not exist.

Eg. $\sum_{n=1}^{\infty} a_n x^n$

$= \left(\frac{x}{2}\right)^1 + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{4}\right)^4 + \cdots$

$\lim \frac{|a_{n+1}|}{|a_n|}$, $\lim |a_n|^{\frac{1}{n}}$ do not exist

Rm: $R = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1} = 2$

Eg 1 (a) $\sum_{n=0}^{\infty} x^n$ conv. on $(-1,1)$
div. elsewhere

(b) $\sum_{n=0}^{\infty} \frac{x^n}{n}$ conv. on $[-1,1)$
div. elsewhere

$a_n = \frac{1}{n} \xrightarrow[\text{Root}]{\text{Ratio}} \rho = 1 \Rightarrow R = 1$

$|x| < 1$. conv, $|x| > 1$. div

$x = 1$. p-series, $p = 1$

$x = -1$ Alternating Series test

(c) $\sum_{n=0}^{\infty} \frac{(-x)^n}{n}$ conv on $(-1, 1]$
div elsewhere.

(d) $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ conv on $[-1, 1]$
div elsewhere.

Algebraic Manipulation of Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$B(x) = \sum_{n=0}^{\infty} b_n x^n$

both conv on $|x| < R$.

Then $A(x) \pm B(x) = ?$

$A(x) \cdot B(x) = ?$

$A(x) / B(x) = ?$

Thm If $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$B(x) = \sum_{n=0}^{\infty} b_n x^n$

both conv (abs.) on $|x| < R$

and $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$

$\left(= \sum_{k=0}^{n} a_k b_{n-k}\right)$

Then $\sum_{n=0}^{\infty} c_n x^n$ conv. abs. and $= A(x)B(x)$

on $|x| < R$

Sol. $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$B(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$

$\Rightarrow A(x) \cdot B(x) = a_0 b_0 + (a_1 b_0) x + (a_2 b_0 + a_1 b_1 + a_0 b_2) x^2 + \cdots$

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$B(x) = \sum_{n=0}^{\infty} b_n x^n$

both converge (abs.) on $|x| < R$

Can we find Power series Representation of $\frac{A(x)}{B(x)}$ if $b_0 \neq 0$?

<u>Sol</u>. <u>If</u> $C(x) = \frac{A(x)}{B(x)} = \sum_{n=0}^{\infty} c_n x^n$

$\Rightarrow A(x) = B(x) \cdot C(x)$

$\Rightarrow a_0 = b_0 c_0 \Rightarrow c_0 = \frac{a_0}{b_0}$

$a_1 = b_0 c_1 + b_1 c_0 \Rightarrow c_1 = \frac{1}{b_0}(\cdots\cdots)$

$a_2 = b_0 c_2 + b_1 c_1 + b_2 c_0 \Rightarrow c_2 = \frac{1}{b_0}(\cdots\cdots)$

$\cdots\cdots$,

Alternatively, we can find C_n more efficiently by long division.

Eg 1. $A(x) = 1$

$$B(x) = 1 + x + \frac{x^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(In fact $B(x) = e^x$)

Find first few terms of $\frac{A(x)}{B(x)}$

$$
\begin{array}{r|l}
 & 1 \quad -1 \quad +\frac{1}{2} \quad -\frac{1}{6} + \cdots \\
\hline
1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots & 1 + 0 + 0 + 0 + \cdots \\
 & 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots \\
\cline{2-2}
 & \quad -1 - \frac{1}{2} - \frac{1}{6} + \cdots \\
 & \quad -1 - 1 - \frac{1}{2} + \cdots \\
\cline{2-2}
 & \qquad \frac{1}{2} + \frac{1}{3} + \cdots \\
 & \qquad \frac{1}{2} + \frac{1}{2} \\
\cline{2-2}
 & \qquad\quad -\frac{1}{6} + \cdots
\end{array}
$$

Ans:

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

$(= e^{-x})$

Rm If $A(x), B(x)$

both conv. (abs.) on $|x|<R$

and $b_0 = B(0) \neq 0$.

$\Rightarrow \exists\, \delta > 0$ such that

$\frac{A(x)}{B(x)}$ computed above

converges on $|x| < \delta$.

Eg2: $A(x) = 1$, $B(x) = \frac{1-x}{1+x^2}$

both converge on $|x| < \infty$

but $\frac{A(x)}{B(x)} = 1 + x + x^2 + x^3 + \cdots$ $(\delta = 1)$

$1 - x^2 + x^4 - x^6 + \cdots$ $(\delta = 1)$

Term by term differentiation

Thm If $f(x)=\sum_{n=0}^{\infty} C_n (x-a)^n$

converges (abs.) on $|x-a|<R$

Then: (1): $f', f'', f''', \ldots$

all exist on $|x-a|<R$

(2). $f'(x)=\sum_{n=1}^{\infty} nC_n (x-a)^{n-1}$

$f''(x)=\sum_{n=2}^{\infty} n(n-1)C_n(x-a)^{n-2}$

$\ldots$

all converge on $|x-a|<R$

Eg 3: $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

$= \frac{1}{1-x}$ on $|x| < 1$

what is the power series representation of $\frac{1}{(1-x)^2}$?

Sol $\frac{1}{(1-x)^2} = f'(x)$

$= 1' + x' + (x^2)' + \cdots + (x^n)' + \cdots$

$= 1 + 2x + \cdots + nx^{n-1} + \cdots$

$= \sum_{n=1}^{\infty} nx^{n-1}$ converges on $|x| < 1$

Rm $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = ?$ Ans. $= \left(\sum_{n=1}^{\infty} nx^{n-1}\right)_{x=\frac{1}{2}}$

$= \left(\frac{d}{dx}\sum_{n=0}^{\infty} x^n\right)_{x=\frac{1}{2}} = \frac{1}{(1-x)^2}\Big|_{x=\frac{1}{2}} = 4$

Eg4. $\sum_{n=1}^{\infty} n^2 x^n = ?$

Sol $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ — ①

$(|x|<1)$

$\frac{d}{dx} \Rightarrow \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$ — ②

$\frac{d}{dx} \Rightarrow \frac{2}{(1-x)^2} = \sum_{\substack{n=2 \\ (n=1)}}^{\infty} n(n-1) x^{n-2}$ — ③

$$n^2 x^n = x^2\left(n(n-1)x^{n-2}\right) + x\left(n x^{n-1}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 x^n = \sum_{\substack{n=1 \\ (n=2)}}^{\infty} n(n-1) x^n + \sum_{n=1}^{\infty} n x^n$$

$$= x^2 \cdot \left(\frac{1}{1-x}\right)'' + x\left(\frac{1}{1-x}\right)'$$

$$= \frac{x+x^2}{(1-x)^3} \quad \text{valid on } |x|<1$$

Remark Term by term differentiation may not be valid for other series

Eg5: $f(x) = \sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$ is not a power series

Since $|a_n| \leq \frac{1}{n^2} \Rightarrow f(x)$ converges on $x \in \mathbb{R}$

But $\sum_{n=1}^{\infty} n a_n = \sum_{n=1}^{\infty} \frac{n!}{n^2} \cos(n!x)$

diverges for any $x \in \mathbb{R}$.

Thm (term by term integration)

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges abs. on $|x-a|<R$

Then $\sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$ also converges on $|x-a|<R$

and $\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C$

In fact

$$\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \int_a^x c_n (t-a)^n\,dt$$
$$= \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

Eg6. Evaluate

$$F(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$$

Sol. Radius of convergence:

Ratio / Root test: $F(x)$ converge if $|x|^2 < 1$

$\Rightarrow R = 1$

On $|x| < 1$, $F'(x) = 1 - x^2 + x^4 - \cdots = \frac{1}{1+x^2}$

$$F(x) = \int_0^x F'(t)\,dt = \int_0^x \frac{1}{1+t^2}\,dt$$

$$= \tan^{-1} x.$$

Note: $\tan^{-1} x$ is defined for all $x \in \mathbb{R}$

but $\neq \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$ if $|x| \geq 1$

Eg: $\ln(1 \pm x)$, $|x| < 1$

$$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \cdots$$

$$\int_0^x \frac{1}{1 \pm t}\,dt = x \mp \frac{x^2}{2} + \frac{x^3}{3} \mp \frac{x^4}{4} + \cdots$$

$$\pm \ln|1 \pm t|\Big|_0^x \underset{|x|<1}{=\!=} \pm \ln(1 \pm t)\Big|_0^x = \pm \ln(1 \pm x)$$

$$\Rightarrow \ln(1 \pm x) = \begin{cases} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \end{cases}$$

$|x| < 1$

Taylor Series:

Question: For a given function $f(x)$ and $a \in \mathbb{R}$, can we always find $a_k \in \mathbb{R}$ and $R > 0$ such that

$$(*) \quad f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

on $|x-a| < R$?

Ans: Not Necessarily.
(only for some f, not all f)

Remark If $a_k \in \mathbb{R}$ and
$R > 0$ do exist, then
we must have $a_k = \frac{f^{(k)}(a)}{k!}$ (✡)
from term by term diff. Thm.
That is, (✡) is the only candidate
and it may or may not work!

Question

If $f^{(k)}(a)$ exist for all $k = 0, 1, 2, \cdots$

Is it necessarily true that (*) holds with (**) for some $R > 0$?

Ans: Not necessarily.

(Counter example below

i.e. $f(x) \neq \sum_{k=0} \frac{f^{(k)}(a)}{k!}(x-a)^k$ for $x \neq a$)

Def: The Taylor Series generated by f at $x=a$

$$T_{f,a}(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

(= Mclaurin Series if $a=0$)

Def The Taylor Polynomial of degree n generated by f at $x=a$: $P_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Eg 1: f is a polynomial.

$$f(x) = a_0 + a_1 x + \cdots + a_5 x^5$$

Find $P_{3,0}(x)$, $P_{5,0}(x)$, $P_{7,0}(x)$, $T_{f,0}(x)$

Ans: $f^{(k)}(0) = k!\, a_k \quad 0 \le k \le 5$

$f^{(l)}(0) = 0$ for $l > 5$

$$\Rightarrow P_{3,0}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$P_{5,0}(x) = P_{7,0}(x) = T_{f,0}(x) = f(x)$$

Rm $P_{3,0}(x) \neq P_{3,1}(x)$, $T_{f,0}(x) = T_{f,1}(x)$

$P_{5,0}(x) = P_{5,1}(x)$, $P_{7,0}(x) = P_{7,1}(x)$

In general, if $f(x)$ is a polynomial of degree n, then

$$P_{m,a}(x) = T_{f,a}(x) = f(x)$$

for all $m \geq n$.

$$\because f(x) = a_0 + a_1 x + \cdots + a_n x^n = P_{n,0}(x)$$

$$= b_0 + b_1(x-a) + \cdots + b_n(x-a)^n = P_{n,a}(x)$$

$$\left(b_k = \frac{f^{(k)}(a)}{k!}\right)$$

$$P_{m,a}(x) = P_{n,a}(x) \text{ if } m > n.$$

$$\therefore P_{m,a}(x) = f(x) = T_{f,a}(x).$$

Eg 2 $f(x)=e^x$. $T_{f,a}(x)=?$

<u>Ans</u>: $f^{(k)}(a)=e^a$

$\therefore T_{f,a}(x)=e^a\sum_{k=0}^{\infty}\frac{(x-a)^k}{k!}$ (就是!)

<u>Remark</u>: from ratio test

$\rho=0 \Rightarrow R$ (for $\sum\frac{(x-a)^k}{k!}$) $=\infty$

$\therefore T_{f,a}(x)$ converges for any $x\in\mathbb{R}$

(In fact, $T_{f,a}(x)=e^x$ for any $x\in\mathbb{R}$
(later))

Eg3: $T_{\cos(x),0}(x)$

Sol $\cos^{(n)}(0) = ?$

$n=0$	$n=1$	$n=2$	$n=3$
$4, 8, \cdots$	$5, 9, \cdots$	$6, 10, \cdots$	$7, 11, \cdots$
$\cos 0$	$-\sin 0$	$-\cos 0$	$\sin 0$
$\parallel$	$\parallel$	$\parallel$	$\parallel$
1	0	-1	0

$$\Rightarrow T_{\cos(x),0}(x)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad (背!)$$

$(= \cos x$ for all $x \in \mathbb{R}$ (later))

Similarly

$$T_{\sin(x),0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} \quad (背!)$$

<u>Eg 4.</u> $T_{\frac{1}{1-x},0}(x) = \,?$

<u>Sol:</u> $f(x) = \frac{1}{1-x} = (1-x)^{-1}$

$f'(x) = +(1-x)^{-2}$, $\quad f'(0) = 1$

$f''(x) = +2(1-x)^{-3}$ $\quad f''(0) = 2!$

$f^{(k)}(x) = k!(1-x)^{-k-1}$ $\quad f'''(0) = 3!$

$$\Rightarrow T_{\frac{1}{1-x},0}(x) = 1 + x + x^2 + x^3 + \cdots$$

Thm A: If $f(x)$ has a power series representation $\left(f(x)=\sum_{k=0}^{\infty} a_k (x-a)^k\right)$ on $|x-a|<R$, $R>0$

$\Rightarrow f^{(k)}(a) = k!\, a_k \quad k \in N$

(Term by Term diff)

$\Rightarrow T_{f,a}(x) = \sum_{k=0}^{\infty} \frac{k!\, a_k}{k!}(x-a)^k = f(x)$

on $|x-a|<R$

$\Rightarrow f(x) = T_{f,a}(x)$ on $|x-a|<R$

Method 2 for Eg 4:

$\because \frac{1}{1-x} = 1+x+x^2+\cdots \quad (R=1>0) \Rightarrow T_{\frac{1}{1-x},0}(x) = 1+x+x^2+\cdots$

In the following example

$f^{(k)}(0) = 0$ for all k

$\Rightarrow T_{f,0}(x) \equiv 0$ but $f(x) \neq 0$
$(x \neq 0)$

<u>Eg5</u> $f(x) = \begin{cases} e^{\frac{-1}{x^2}}, & x \neq 0 \\ 0, & x = 0, \end{cases}$ $T_{f,0}(x) = ?$

<u>Ans</u>: $f(0) = 0$

$$f'(0) = \lim_{h \to 0} \frac{e^{\frac{-1}{h^2}} - 0}{h - 0} \quad \left(\begin{array}{c} \underline{\text{Not}} \text{ always} \\ = \lim_{h \to 0} f'(h) \end{array} \right)$$

$$\left(= \text{"}\frac{0}{0}\text{"}\right) = \lim_{h \to 0} \frac{\left(\frac{1}{h}\right)}{e^{\frac{1}{h^2}}} \quad \left(= \text{"}\frac{\infty}{\infty}\text{"}\right)$$

<u>l'Hôpital</u> $\lim_{h \to 0} \frac{-h^{-2}}{-2h^{-3} e^{\frac{1}{h^2}}} = \lim_{h \to 0} \frac{h}{2 e^{\frac{1}{h^2}}} = 0$

$(**)$

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h - 0}$$

$$\overset{(**)}{=} \lim_{h \to 0} \frac{2h^{-3} e^{-\frac{1}{h^2}}}{h - 0}$$

$$= \lim_{h \to 0} \frac{2 e^{-\frac{1}{h^2}}}{h^4}$$

$$= \lim_{h \to 0} \frac{2h^{-4}}{e^{-\frac{1}{h^2}}}$$

$$\overset{\text{L'Hôpital}}{=} \text{(homework)}$$

$$= 0$$

In fact, $f^{(k)}(0) = 0$

for all $k \in \mathbb{N}$

$$T_{f,a}(x) \overset{\text{def}}{=\!=} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

(1) For what values of x does it converge? i.e. $R = ?$

(2) When it converges, does it equal $f(x)$?

Eg: $f(x) = \begin{cases} 0 & x = 0 \\ e^{-\frac{1}{x^2}} & x \neq 0 \end{cases}$

$T_{f,a}(x) = 0 \neq f(x)$ for $x \neq 0$

Thm Taylor's Thm

If $f, f', f'', \cdots$

all exist on $|x-a|<\delta$

Then, for any $n \in \mathbb{N}$

$$f(x) = P_n(x) + R_n(x) \quad \cdots (*)$$

on $|x-a|<\delta$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x

Since $\overline{T}_{f,a}(x) = \lim\limits_{n\to\infty} P_n(x)$

$\therefore$ $\overline{T}_{f,a}(x) = f(x) \iff \lim\limits_{n\to\infty} R_n(x) = 0$

Corollary: If $|f^{(n+1)}(c)| \leq M$

for all c between a and x

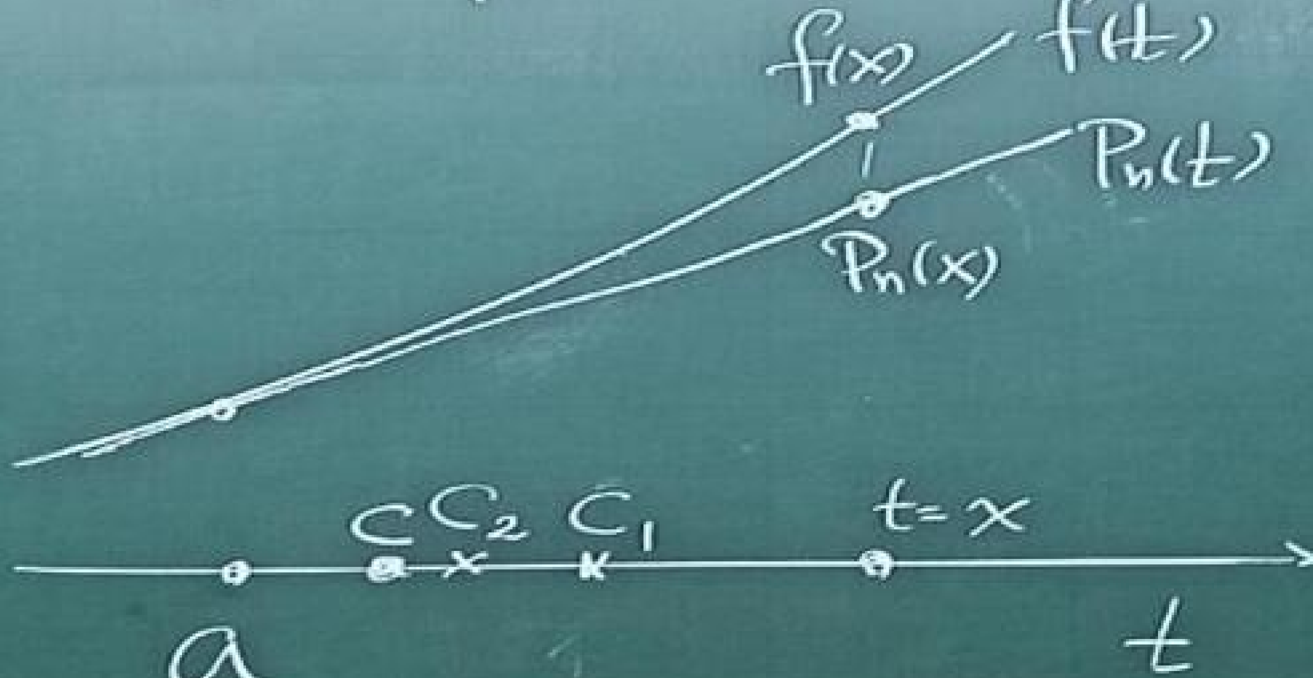
and all $n \in \mathbb{N}$

Then $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$

$\Rightarrow \lim\limits_{n\to\infty} R_n(x) = 0$

$\Rightarrow \overline{T}_{f,a}(x) = f(x)$

Proof of Taylor's Thm



We now write (for fixed x)

$$f(x) = P_n(x) + K(x-a)^{n+1}$$

$$\left(\text{i.e. } K = \frac{f(x)-P_n(x)}{(x-a)^{n+1}}\right)$$

We want to show $K = \dfrac{f^{(n+1)}(c)}{(n+1)!}$

Define $F(t) = f(t) - \left(P_n(t) + K(t-a)^{n+1}\right)$

$$F(a) = f(a) - (P_n(a) + 0) = 0$$

$$F(x) = 0$$

$\overset{\text{M.V.T.}}{\Longrightarrow}$ $F'(c_1)=0$ for some c_1 between a, x

$$F'(a)=f'(a)-(P_n'(a)+0)=0$$

$$\because\ P_n'(a)=\frac{f'(a)}{1}+\frac{2f''(a)}{2!}(a-a)+\frac{3f'''(a)}{3!}(a-a)^2+\cdots\cdots$$

$\overset{\text{MVT for } F'}{\Longrightarrow}$ $F''(c_2)=0$ for some c_2 between a and c_1

$\vdots$

$F^{(n+1)}(c)=0$ for some c between a and c_n

$$\parallel$$

$$f^{(n+1)}(c)-(0+K\cdot n!)=0$$

$$\Rightarrow K=\frac{f^{(n+1)}(c)}{(n+1)!} \qquad \text{———} \ \square$$

From the proof, it is easy to see that $c\ (=c_{n+1})$ depends on c

Eg 1 $f(x)=e^x$

$$\overline{T}_{f,a}(x)=\sum_{k=0}^{\infty}\frac{e^a}{k!}(x-a)^k$$

$$\xrightarrow{\text{Ratio Test}} \rho = 0 \Rightarrow R=\infty$$

$\therefore \overline{T}_{f,a}(x)$ converges for all $x\in\mathbb{R}$

Moreover $R_n(x)=\dfrac{e^c}{(n+1)!}(x-a)^{n+1}$

c is between a and x

$$a<x \Longrightarrow c<x$$
$$x<a \Longrightarrow c<a$$

$$\therefore e^c \le \underbrace{\max(e^a, e^x)}_{\text{independent of } n}$$

$$\therefore \lim_{n\to\infty} R_n(x)=0 \Rightarrow \overline{T}_{f,a}(x)\underset{x\in\mathbb{R}}{=}e^x$$

Eg 2. $f(x) = \sin x,\ a = 0$

$$T_{\sin x, 0} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

From Ratio Test $\Rightarrow R = \infty$

$\therefore T_{\sin x, 0}$ converges for $x \in \mathbb{R}$

Secondly $R_n(x) = \frac{\sin^{(n+1)}(c)}{(n+1)!} x^{n+1}$

$$\left|\sin^{(n+1)}(c)\right| = \left|\left(\frac{d^{n+1}}{dx^{n+1}} \sin x\right)_{x=c}\right| \leq 1$$

$$\therefore |R_n| \leq \frac{|x|^{n+1}}{(n+1)!} \xrightarrow[n \to \infty]{} 0$$

$$T_{\sin x, 0}(x) = \sin x,\ \forall x \in \mathbb{R}$$

Rm(1). Similarly

$$\cos x = \overline{T}_{\cos x, 0}(x)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots$$

for any $x \in \mathbb{R}$

(2) $\overline{T}_{\sin x, a}(x) = \sum_{k=0}^{\infty} \frac{\sin^{(k)}(a)}{k!}(x-a)^k$

$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \left| \frac{\sin^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} \right|$$

$$= 0 \quad \forall\, x \in \mathbb{R}$$

$\Longrightarrow \overline{T}_{\sin x, a}(x)$ converges $\forall\, x \in \mathbb{R}$

and $= \sin x$

Similarly for $\overline{T}_{\cos x, a}(x)$

Eg3. Find first few terms of $T_{f,0}(x)$ where $f(x) = e^x \cos x$

Sol. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\cdots$

both holds for all $x \in \mathbb{R}$

From Multiplication of Power Series

$(e^x \cdot \cos x)$ has Power Series Representation

and $= 1 + x + \left(\frac{1}{2!} - \frac{1}{2!}\right)x^2 + \frac{x^3}{3!}$

$+ \left(\frac{1}{4!} - \frac{1}{2!}\frac{1}{2!}\right)x^4 + \cdots$

$\xRightarrow{\text{Thm A}} = T_{e^x \cos x, 0}(x)$