Alternating Series (Leibnitz test)

If (1) $u_n > 0$

(2) $u_n \geqq u_{n+1}$ (for all $n \geqq N$)

(3) $\lim_{n \to \infty} u_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges

Eg 1: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges

Sol $u_n = \frac{1}{n} > 0$.

$u_n > u_{n+1}$, $\lim u_n = 0$

$\therefore$ Leibnitz test $\Rightarrow$ converges

Def (1) $\sum a_n$ converges absolutely
if $\sum |a_n| < \infty$

(2) $\sum a_n$ converges conditionally
if $\begin{cases} \sum a_n \text{ converges} \\ \sum |a_n| = \infty \end{cases}$

Eg2 (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ $p > 0$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{conv.} & p > 1 \\ \text{div} & 0 < p \leq 1 \end{cases}$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ converges $\begin{matrix} \text{abs.} & p > 1 \\ \text{cond.} & 0 < p \leq 1 \end{matrix}$

Eg 3 $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{\ln n}}$ conv abs? cond?

Sol: $\sum (-1)^{n+1} u_n$ converges (Leibnitz Test)

Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ converge?

Integral test:

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \int_{x=2}^{\infty} \frac{1}{\sqrt{\ln x}}\,d\ln x$$

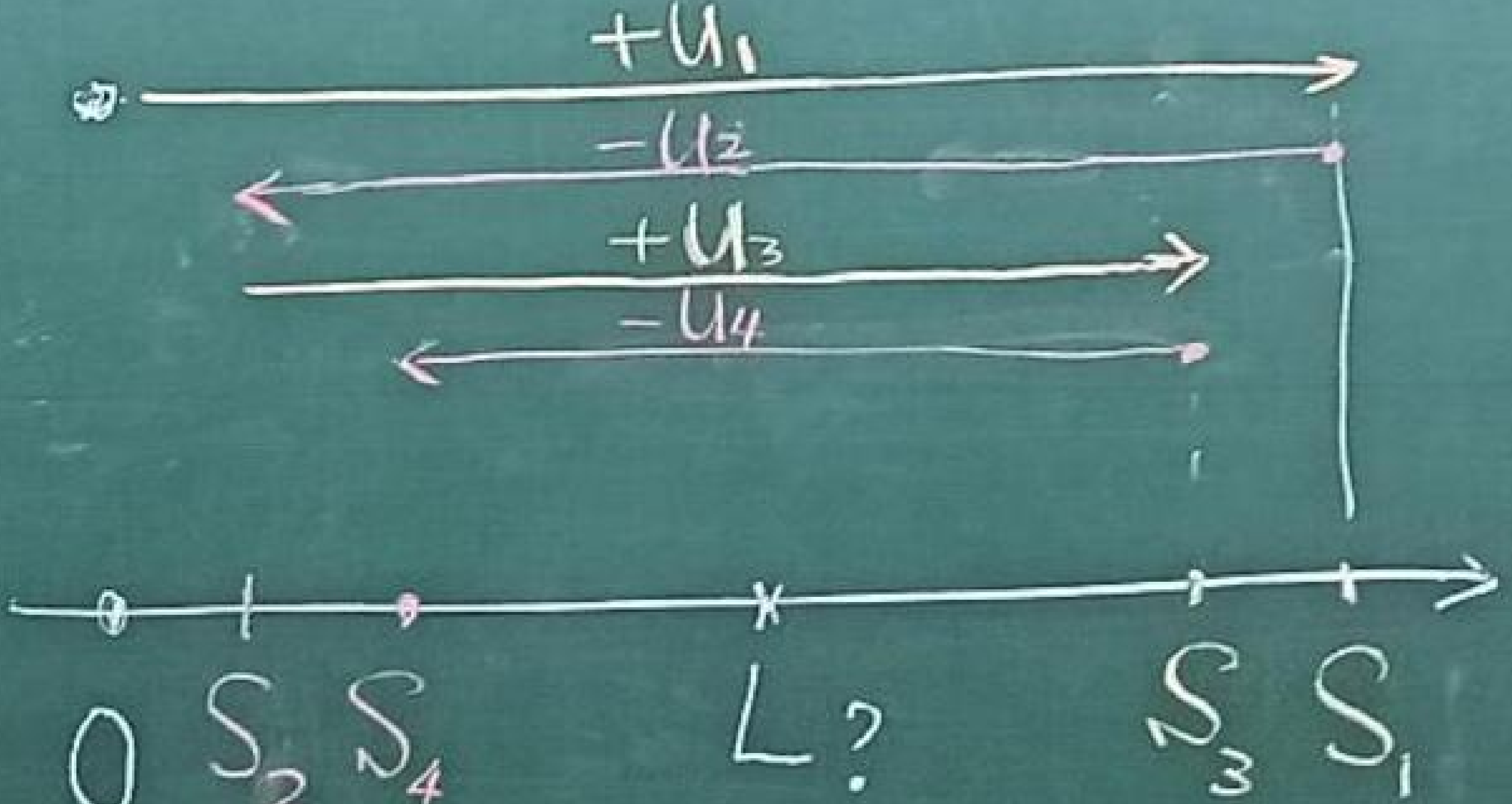
$$= \int_{y=\ln 2}^{\infty} \frac{1}{\sqrt{y}}\,dy \quad (y = \ln x)$$

$= "p = \frac{1}{2}" =$ divergent.

Ans: $\displaystyle\sum \frac{(-1)^{n+1}}{n\sqrt{\ln n}}$ converges cond.

Pf of Leibnitz test

Let $S_n = \sum_{k=1}^{n} (-1)^{k+1} u_k$



$\Rightarrow 0 < S_2 < S_4 < \cdots < S_3 < S_1$

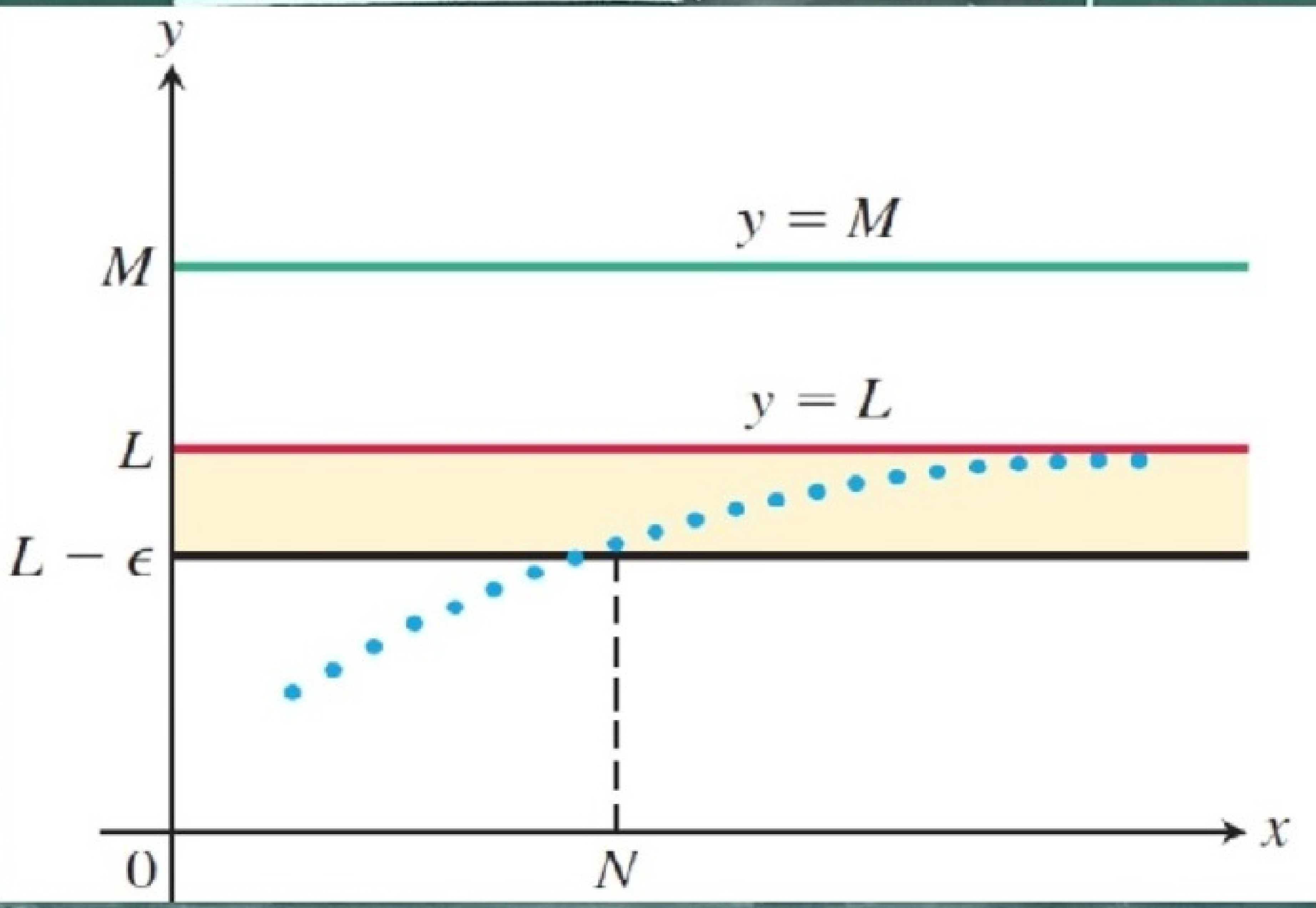
$\therefore \{S_2, S_4, \dots, S_{2k}, \dots\}$

is an <u>increasing</u> sequence and <u>bounded above</u> ($S_{2k} < S_1$)

Thm 6 (Section 10.1)

If $a_1 \le a_2 \le \cdots \le a_n \le \cdots\cdots \le M$ for some $M \in \mathbb{R}$. Then $\lim_{n\to\infty} a_n$ exists

pf: Let L = Smallest M such that $a_n \le M$ for all n



Then (a) $a_n \leq L$ for all n.

(b). For any $\varepsilon > 0$. some $a_N > L - \varepsilon$

It follows that

$$L \geq a_n \geq a_N > L - \varepsilon \text{ for all } n \geq N$$

$$\Rightarrow |a_n - L| < \varepsilon \ \forall n \geq N$$

$$\Rightarrow L = \lim_{n \to \infty} a_n \text{ exists}$$

Similarly if $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq M$

From the Monotone Sequence Thm (Section 10.1. Thm 6)

$$\lim_{k\to\infty} S_{2k} = L \text{ for some } L \in \mathbb{R}$$

Moreover $S_{2k+1} = S_{2k} + u_{2k+1} \xrightarrow[k\to\infty]{} L + 0$

$$\therefore \lim_{n\to\infty} S_n = L$$

Remark: (1) We assumed for simplicity $u_1 \geq u_2 \geq u_3 \geq \cdots\cdots$ (i.e. decreasing for all n)

(2) $S_{2k} < L < S_{2l-1}$ $\forall k, l \in \mathbb{N}$

$\Rightarrow$ $0 < L - S_{2k} < u_{2k+1}$

$0 < S_{2l-1} - L < u_{2l}$

(error estimate of partial sum)

Def. Power Series

$$\sum_{n=0}^{\infty} C_n(x-a)^n \overset{\text{def}}{=\!=} C_0 + \sum_{n=1}^{\infty} C_n(x-a)^n$$

It is a function of x

a = center (fixed). C_n = Coefficients

Question: When "a" and "C_n" are given, for what values of x is the power series convergent?

Eg 1. $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n$. (Geometric Series)

$r = \frac{-(x-2)}{2}$ (div if $|r| \geq 1$)

Power Series converges (abs) $\Leftrightarrow |r| < 1 \Leftrightarrow 0 < x < 4$

Eg2 $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$(0! \overset{\text{def}}{=} 1.\ x^0 \overset{\text{def}}{=} 1)$

Sol: Ratio test:

$|u_n| = \frac{|x|^n}{n!}$

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x|}{n} = 0$$

$\Rightarrow$ For any $x \in \mathbb{R}$

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges (absolutely)

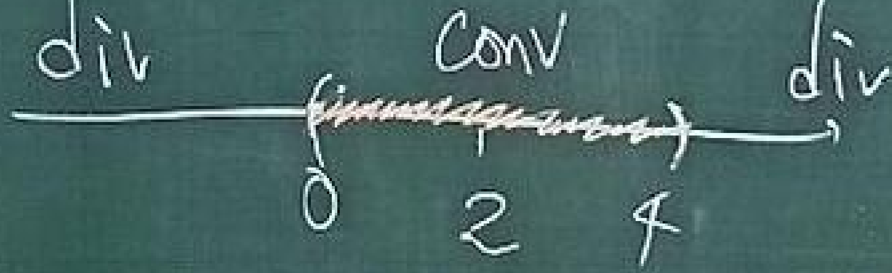
Eg3 $\sum_{n=0}^{\infty} n!\,x^n$

Ratio Test:

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} n|x| = \begin{cases} 0, & x=0 \\ \infty, & x\neq 0 \end{cases}$$

It converges at $x=0$ only and diverges elsewhere.

Eg1



Eg2

Eg3