Def (1) $\sum a_n$ converges absolutely
if $\sum |a_n| < \infty$

Thm: The Absolute Convergence Test

Thm $\sum_{n=1}^{\infty} |a_n| < \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ conv.

pf: $-|a_n| \le a_n \le |a_n|$, $0 \le a_n + |a_n| \le 2|a_n|$

$\sum |a_n| < \infty \Rightarrow \sum 2|a_n| < \infty \Rightarrow \sum a_n + |a_n| < \infty$

$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n| = \text{conv} - \text{conv} = \text{conv}.$

Ratio Test and Root Test.

Thm (Ratio Test)

If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho$,

ⓐ $0 \le \rho < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

ⓑ $\rho > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ div.

ⓒ $\rho = 1 \Rightarrow$ inconclusive

Thm (Root Test)

If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho$

ⓐ $0 \le \rho < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

ⓑ $\rho > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ div.

ⓒ $\rho = 1 \Rightarrow$ inconclusive.

Eg 1 $\begin{cases} \sum \frac{1}{n} = \infty & (\rho = 1) \\ \sum \frac{1}{n^2} < \infty & (\rho = 1) \end{cases}$

case ⓒ for both ratio test and root test.

∞

Eg2 $a_n = (-1)^{n+1}\frac{1}{n}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$$\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k-1} - \frac{1}{2k} + \cdots$$

$$= \sum_{k=1}^{\infty}\left(\frac{1}{2k-1} - \frac{1}{2k}\right)$$

$$= \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \quad \text{conv}$$

$\left(\because \text{Compare with } \sum_{k=1}^{\infty} \frac{1}{k^2}\right)$

$\because \left(\sum |a_n| = \infty \not\Rightarrow \sum a_n \text{ div}\right)$

Eg 3

ⓐ $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

Sol: Ratio Test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim \frac{2^{n+1}+5}{3\cdot(2^n+5)}$$

$$= \lim_{n\to\infty} \frac{2+5\cdot 2^{-n}}{3\cdot(1+5\cdot 2^{-n})} = \frac{2}{3}$$

$\rho < 1 \Rightarrow$ convergent.

Root test:

$$\lim |a_n|^{\frac{1}{n}} = \lim \left(\frac{2^n(1+5\cdot 2^{-n})}{3^n}\right)^{\frac{1}{n}}$$

$$= \frac{2}{3} < 1$$

(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

Sol. Ratio Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(2n+2)!}{(n+1)!\,(n+1)!}}{\dfrac{(2n)!}{n!\,n!}}$$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} = 4 > 1$$

Ratio test $\Rightarrow$ divergent

(c) $\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$

Sol: Ratio Test

$\rho = 1$ (See (b))

Ratio Test is inconclusive

However,

$$\frac{a_{n+1}}{a_n} \overset{(b)}{=} \frac{4(n+1)(n+1)}{(2n+1)(2n+2)}$$

$$= \frac{(2n+2)\cancel{(2n+2)}}{(2n+1)\cancel{(2n+2)}} > 1$$

$\Rightarrow a_{n+1} > a_n > 0$

$\Rightarrow \lim_{n\to\infty} a_n \neq 0$, $\sum a_n$ div!

(d) $\sum_{n=1}^{\infty} \frac{n^2}{2^n} \xrightarrow[\text{Root}]{\text{Ratio}} \rho = \frac{1}{2}$. conv.

(e) $\sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right)^n \xrightarrow{\text{Root}} \rho = 0$. conv.

(f) $a_n = \begin{cases} \frac{n}{2^n} & n \text{ is odd} \\ \frac{1}{2^n} & n \text{ is even} \end{cases}$

$= \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{1}{16}, \frac{5}{32}, \frac{1}{64}, \frac{7}{128}, \frac{1}{256}, \cdots$

$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{2} \cdot \frac{1}{n} & n \text{ is odd.} \\ \frac{1}{2} (n+1) & n \text{ is even} \end{cases}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist: inconclusive.

However $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \frac{1}{2} \Rightarrow$ convergent.

Proof (of The Ratio Test
The Root test is similar)

(1) $0 \leq \rho < 1$, $\left(\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \right)$

Take $r = \frac{1+\rho}{2} < 1$

Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$

Take $\varepsilon = r - \rho > 0$, there exists a corresponding $N \in \mathbb{N}$ such that

"$n > N \Longrightarrow \left| \left| \frac{a_{n+1}}{a_n} \right| - \rho \right| < \varepsilon$"

$\left(\Rightarrow \left| \frac{a_{n+1}}{a_n} \right| < \underbrace{\rho + \varepsilon}_{= r} \right)$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

$$= \sum_{n=1}^{N} |a_n| + |a_{N+1}| + |a_{N+2}| + \cdots$$

$$\leq (\quad " \quad) + |a_N| r + |a_N| r^2 + \cdots\cdots$$

$$= \left(\sum_{n=1}^{N} |a_n| \right) + \text{convergent Geometric Series} < \infty$$

(2) $1 < \rho$, define $r = \frac{1+\rho}{2} > 1$

let $\varepsilon = \rho - r > 0 \Rightarrow \exists N \in \mathbb{N}$

such that

"$n > N \Rightarrow \left| \left| \frac{a_{n+1}}{a_n} \right| - \rho \right| < \varepsilon$"

$\Rightarrow \left| \frac{a_{n+1}}{a_n} \right| > \rho - \varepsilon = r > 1$

$\Rightarrow \lim |a_n| \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges