$$\text{Def (1)} \sum a_n \text{ Converges } \underline{\text{absolutely}}$$
$$\text{if } \sum |a_n| < \infty$$

Thm: The Absolute Convergence Test

$$\underline{\text{Thm}} \quad \sum_{n=1}^{\infty} |a_n| < \infty \Rightarrow \sum_{n=1}^{\infty} a_n \text{ conv.}$$

$$\underline{\text{pf}}: \ -|a_n| \le a_n \le |a_n|, \ 0 \le a_n + |a_n| \le 2|a_n|$$

$$\sum |a_n| < \infty \Rightarrow \sum 2|a_n| < \infty \Rightarrow \sum a_n + |a_n| < \infty$$

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n| = \text{conv} - \text{conv} = \text{conv}$$

Ratio Test and Root Test

Thm (Ratio Test)

If $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = \rho$,

(a) $0 \le \rho < 1 \Rightarrow \sum\limits_{n=1}^{\infty} |a_n| < \infty$

(b) $\rho > 1 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ div.

(c) $\rho = 1 \Rightarrow$ inconclusive

Thm (Root Test)

If $\lim\limits_{n \to \infty} |a_n|^{\frac{1}{n}} = \rho$

(a) $0 \le \rho < 1 \Rightarrow \sum\limits_{n=1}^{\infty} |a_n| < \infty$

(b) $\rho > 1 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ div.

(c) $\rho = 1 \Rightarrow$ inconclusive.

Eg 1 $\begin{cases} \sum \dfrac{1}{n} = \infty & (\rho = 1) \\ \sum \dfrac{1}{n^2} < \infty & (\rho = 1) \end{cases}$

case (c) for both ratio test and root test.

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$$\text{Eg2 } a_n = (-1)^{n+1} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$$\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$
$$+ \cdots + \frac{1}{2k-1} - \frac{1}{2k} + \cdots$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k} \right)$$

$$= \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \qquad \text{Conv}$$

$$\left(\therefore \text{Compare with } \sum_{k=1}^{\infty} \frac{1}{k^2} \right)$$
$$\left(\therefore \left(\sum |a_n| = \infty \not\Rightarrow \sum a_n \text{ div} \right) \right)$$

Eg 3

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n}$

Sol: Ratio Test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim \frac{2^{n+1} + 5}{3 \cdot (2^n + 5)}$$

$$= \lim_{n\to\infty} \frac{2 + 5 \cdot 2^{-n}}{3 \cdot (1 + 5 \cdot 2^{-n})} = \frac{2}{3}$$

$g < 1 \Rightarrow$ convergent.

Root test:

$$\lim |a_n|^{\frac{1}{n}} = \lim \left(\frac{2^n (1 + 5 \cdot 2^{-n})}{3^n} \right)^{\frac{1}{n}}$$

$$= \frac{2}{3} < 1$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

Sol. Ratio Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(2n+2)!}{(n+1)!\,(n+1)!}}{\dfrac{(2n)!}{n!\,n!}}$$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} = 4 > 1$$

Ratio test $\Rightarrow$ divergent

(c) $\displaystyle\sum_{n=9}^{\infty} \frac{4^n n! \, n!}{(2n)!}$

Sol: Ratio Test

$$\rho = 1 \quad (\text{see } \textcircled{b}).$$

Ratio Test is inconclusive

However,

$$\frac{a_{n+1}}{a_n} \stackrel{\textcircled{b}}{=} \frac{4(n+1)(n+1)}{(2n+1)(2n+2)}$$

$$= \frac{(2n+2)(2n+2)}{(2n+1)(2n+2)} > 1$$

$$\Rightarrow a_{n+1} > a_n > 0$$

$$\Rightarrow \lim_{n\to\infty} a_n \neq 0, \quad \sum a_n \text{ div!}$$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n} \xrightarrow[\text{Root}]{\text{Ratio}} \rho = \frac{1}{2}$ conv.

(e) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right)^n \xrightarrow{\text{Root}} \rho = 0$ conv.

(f) $a_n = \begin{cases} \dfrac{n}{2^n} & n \text{ is odd} \\[3mm] \dfrac{1}{2^n} & n \text{ is even} \end{cases}$

$= \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{3}{8}, \dfrac{1}{16}, \dfrac{5}{32}, \dfrac{1}{64}, \dfrac{7}{128}, \dfrac{1}{256}, \cdots$

$\dfrac{a_{n+1}}{a_n} = \begin{cases} \dfrac{1}{2} \cdot \dfrac{1}{n} & n \text{ is odd} \\[3mm] \dfrac{1}{2}(n+1) & n \text{ is even} \end{cases}$

$\displaystyle\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ does not exist : inconclusive.

However $\displaystyle\lim_{n\to\infty} a_n^{\frac{1}{n}} = \dfrac{1}{2} \implies$ convergent.

proof (of The Ratio Test. The Root test is similar)

(1) $0 \leq \rho < 1$, $\left(\rho = \lim\limits_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \right)$

Take $r = \frac{1+\rho}{2} < 1$

Since $\lim\limits_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$

Take $\varepsilon = r - \rho > 0$, there exists a corresponding $N \in \mathbb{N}$ such that

"$n > N \implies \left| \left| \frac{a_{n+1}}{a_n} \right| - \rho \right| < \varepsilon$"

$\left(\implies \left| \frac{a_{n+1}}{a_n} \right| < \rho + \varepsilon \atop = r \right)$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

$$= \sum_{n=1}^{N} |a_n| + |a_{N+1}| + |a_{N+2}| + \cdots$$

$$\leq (\quad '' \quad) + |a_N| r + |a_N| r^2$$
$$+ \cdots \cdots$$

$$= \left(\sum_{n=1}^{N} |a_n| \right) + \text{convergent} < \infty$$
$$\text{Geometric Series}$$

(2) $1 < \rho$, define $r = \frac{1+\rho}{2} > 1$

let $\varepsilon = \rho - r > 0 \Rightarrow \exists N \in \mathbb{N}$

such that

$$\text{"} n > N \Rightarrow \left| \left| \frac{a_{N+1}}{a_N} \right| - \rho \right| < \varepsilon \text{"}$$

$$\Rightarrow \left| \frac{a_{n+1}}{a_n} \right| > \rho - \varepsilon = r > 1, \ \lim |a_n| \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$