The Integral Test

Thm If $a_n > 0$, $a_n = f(n)$
where $f(x)$ is positive,
continuous and decreasing
for all $x \geq N$, $N \in \mathbb{N}$ $\Big\}$ (*)

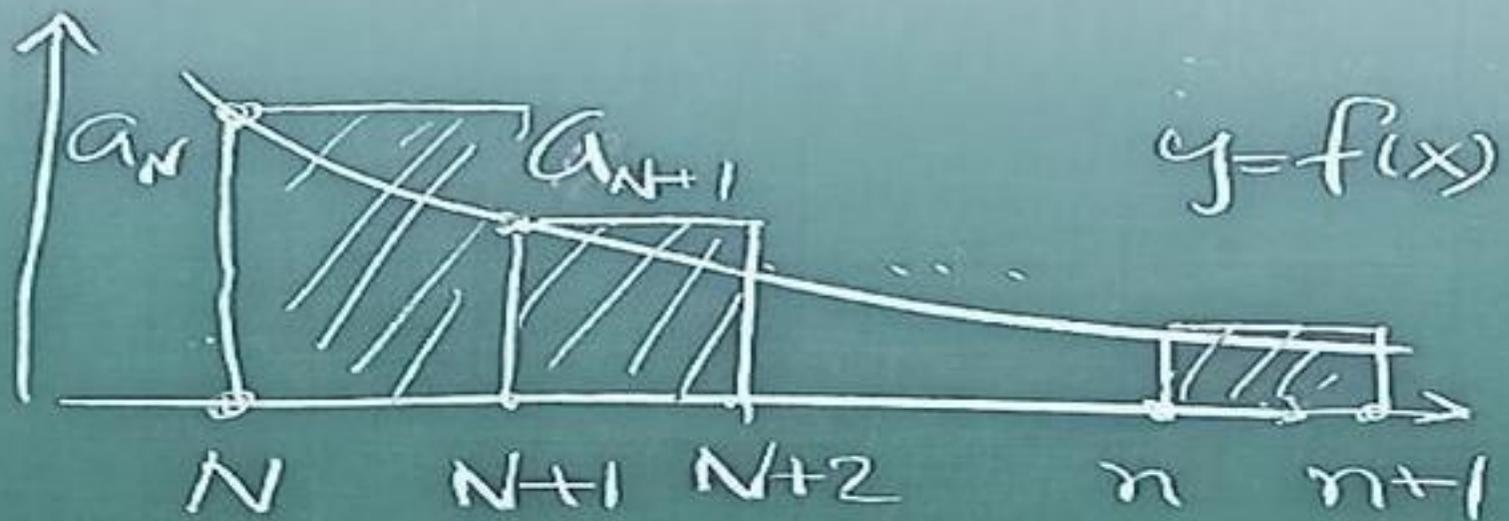Then $\displaystyle\sum_{n=N}^{\infty} a_n < \infty \iff \int_N^{\infty} f(x)\, dx < \infty$

$\Big(a_n > 0, \displaystyle\sum_{n=N}^{\infty} a_n < \infty \iff \sum_{n=N}^{\infty} a_n \text{ conv.}\Big)$

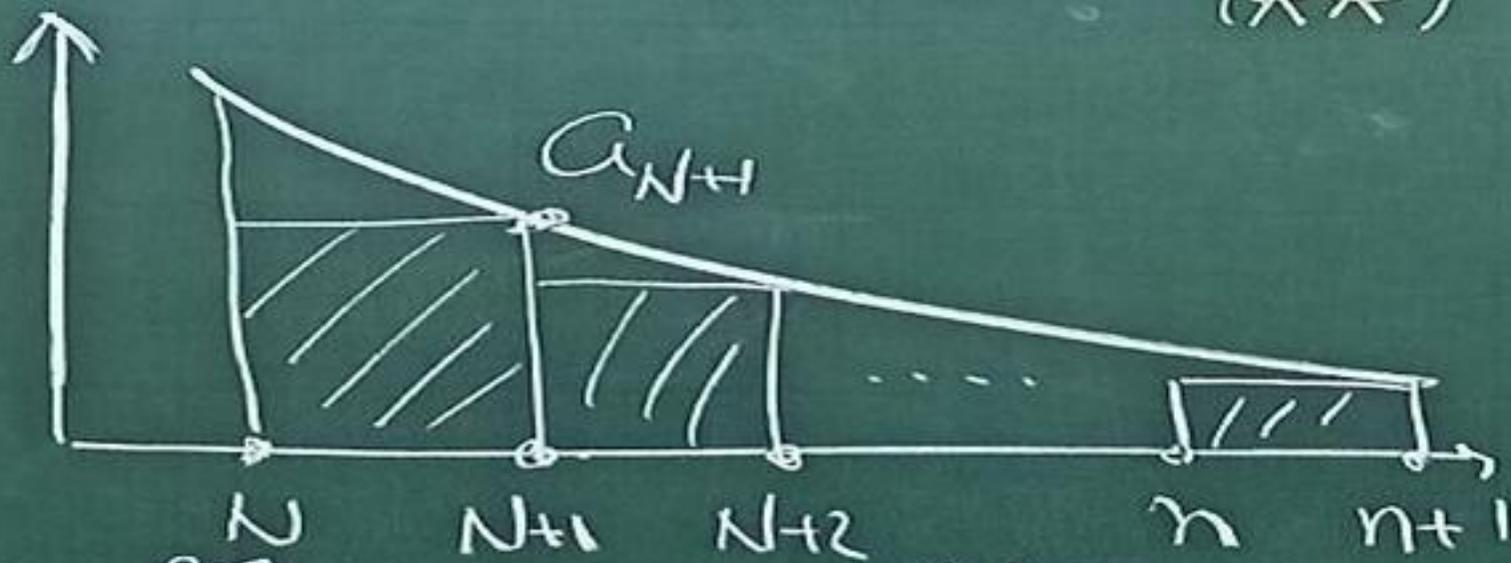**Eg.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges?

Sol $p \leq 0 \implies \displaystyle\lim_{n \to \infty} \frac{1}{n^p} \neq 0, \quad \sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$

$p > 0 \implies$ Compare with $\displaystyle\int_1^{\infty} \frac{1}{x^p}\, dx \begin{cases} < \infty & p > 1 \\ = \infty & 0 < p \leq 1 \end{cases}$

pf.

$$\Rightarrow \sum_{n=N}^{\infty} a_n \geq \int_N^{\infty} f(x)\, dx \qquad (**)$$

$$\sum_{n=N+1}^{\infty} a_n \leq \int_N^{\infty} f(x)\, dx$$

$$(***)$$

$$\int_N^\infty f(x)\,dx \le \sum_{n=N}^\infty a_n \le a_N + \int_N^\infty f(x)\,dx \qquad (**)$$

(a_N is a fixed number)

$$\therefore \sum_{n=N}^\infty f(x)\,dx < \infty \iff \sum_{n=N}^\infty a_n < \infty$$

$\underline{Rm}$. If $(*)$ holds, and

$$\sum_{n=N}^\infty a_n = L < \infty, \text{ then}$$

we can estimate M $(M > N)$

$$R_M = \sum_{n=N}^\infty a_n - \sum_{n=N}^M a_n \left(= \sum_{n=M+1}^\infty a_n \right)$$

$$\int_{M+1}^\infty f(x)\,dx \le \sum_{n=M+1}^\infty a_n \le \int_M^\infty f(x)\,dx$$

$(N \to M+1$ in $(**))$ $\Big\downarrow$ $(N \to M$ in $(***))$

$$= R_M$$

Eg1. $\sum\limits_{n=1}^{\infty} n^{-p} \begin{cases} = \infty, & 0 < p \le 1 \\ < \infty, & p > 1 \end{cases}$

Eg2. $\sum\limits_{n=0}^{\infty} \dfrac{1}{1+n^2} < \infty$

Sol: $\displaystyle\int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{b \to \infty} \int_0^b \frac{1}{1+x^2} dx$

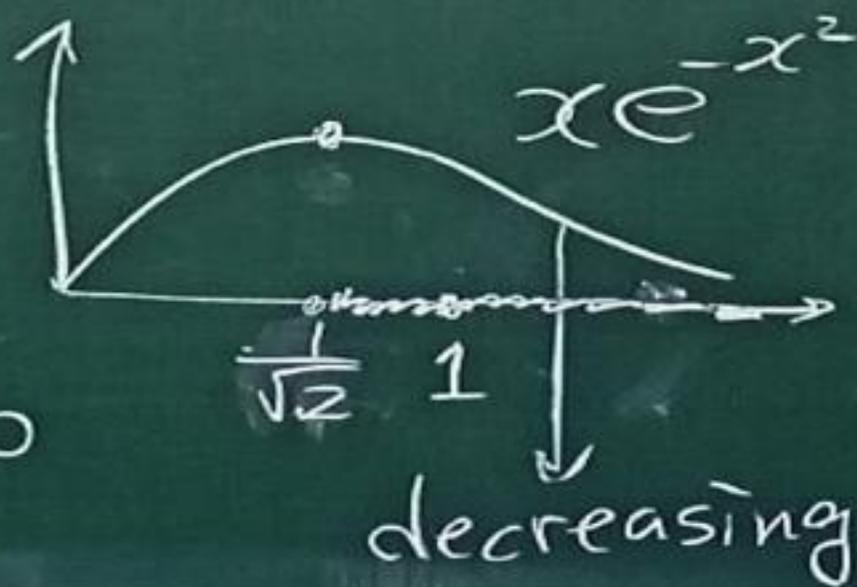$= \displaystyle\lim_{b \to \infty} \tan^{-1} b - \tan^{-1} 0 = \frac{\pi}{2} < \infty$

Eg3. $\sum\limits_{n=1}^{\infty} n e^{-n^2} < \infty$

Sol $\displaystyle\int_1^{\infty} x e^{-x^2} dx$

$= -\dfrac{1}{2} e^{-x^2} \Big|_1^{\infty} < \infty$

$\Rightarrow \sum\limits_{n=1}^{\infty} n e^{-n^2} < \infty$

$x e^{-x^2}$

$\dfrac{1}{\sqrt{2}}$ 1

decreasing

$$\text{Eg4} \quad \sum_{n=2}^{\infty} \frac{1}{2^{\ln n}} \leq \infty \ ?$$

$$\underline{\text{Sol}}: \quad \int_{2}^{\infty} \frac{1}{2^{\ln x}} \, dx = \ ?$$

$$y = \ln x, \quad x = e^{y}, \quad dx = e^{y} dy$$

$$= \int_{y=\ln 2}^{\infty} 2^{-y} e^{y} \, dy$$

$$= \int_{\ln 2}^{\infty} \left(\frac{e}{2}\right)^{y} dy = \infty$$
$$\frac{e}{2} > 1$$

$$\therefore \sum_{n=2}^{\infty} \frac{1}{2^{\ln n}} = \infty$$

Direct Comparison Test

If $0 < d_n \le a_n \le c_n$

for all $n > N$, then

① $\displaystyle\sum_{n=1}^{\infty} c_n < \infty \implies \sum_{n=1}^{\infty} a_n < \infty$

② $\displaystyle\sum_{n=1}^{\infty} d_n = \infty \implies \sum_{n=1}^{\infty} a_n = \infty$

Eg5. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$

Sol $\dfrac{5}{5n-1} > \dfrac{1}{n}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$\implies \displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1} = \infty$

Limit Comparison Test $\left(\sum_{n=1}^{\infty} a_n \overset{?}{<} \infty\right)$

If $a_n > 0$, $b_n > 0$ $\forall n \geq N$

(i) If $\lim_{n \to \infty} \dfrac{a_n}{b_n} = C$, $0 < C < \infty$

Then $\sum_{n=1}^{\infty} b_n \overset{<\infty}{(=\infty)} \Leftrightarrow \sum_{n=1}^{\infty} a_n \overset{<\infty}{(=\infty)}$

(ii) If $\lim_{n \to \infty} \dfrac{a_n}{b_n} = 0$

Then $\sum_{n=1}^{\infty} b_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$

(iii) If $\lim_{n \to \infty} \dfrac{a_n}{b_n} = +\infty$

Then $\sum_{n=1}^{\infty} b_n = \infty \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$

Eg 6 $\sum\limits_{n=1}^{\infty} \dfrac{5}{5n+1}$

Sol $\dfrac{5}{5n+1} < \dfrac{1}{n}$

∴ Direct Comparison Test is inconclusive,

Limit Comparison Test

$$\lim\limits_{n\to\infty} \dfrac{\frac{5}{5n+1}}{\frac{1}{n}} = 1$$

$$\sum\limits_{n=1}^{\infty} \dfrac{1}{n} = \infty \Rightarrow \sum\limits_{n=1}^{\infty} \dfrac{5}{5n+1} = \infty$$

Eg 7. $\sum\limits_{n=0}^{\infty} \dfrac{1}{n!} < \infty \quad (0! = 1)$

Sol: $1 + \dfrac{1}{1} + \dfrac{1}{1\cdot 2} + \dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots$

$< 1 + \dfrac{1}{1} + \dfrac{1}{1\cdot 2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots < \infty$

Eg 8.

	$\sum a_n$	$\sum b_n$	Thm	Result
(a)	$\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n+1)^2}$	$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$	11 (i) $(c=1)$	div.
(b)	$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$	$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$	11 (ii) $(c=1)$	conv.
(c)	$\displaystyle\sum_{n=1}^{\infty} \frac{1+n\ln n}{n^2+5}$	$\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$	11 (ii) $(c=1)$	div.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} \begin{array}{c} < \infty \\ = \infty \end{array} ?$$

Thm 10: Direct Comparison

Thm 11: Limit Comparison

Method 1. $\displaystyle\sum \frac{\ln n}{n} > \sum \frac{1}{n} = \infty$

Method 2: $\displaystyle\int_3^{\infty} \frac{\ln x}{x}\, dx = \infty \Rightarrow \sum_{n=3}^{\infty} \frac{\ln n}{n} = \infty$

($\frac{\ln x}{x}$ is decreasing for $x > e = 2.718...$)

	Σa_n	Σb_n	Thm	Result

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ $\underset{(C=1)}{11\,(i)}$ $\underline{div}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ $\underset{(C=1)}{11\,(i)}$ $\underline{conv}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{\frac{3}{2}}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^{0.2}}{n^{\frac{3}{2}}}$ $\underset{(C=0)}{11\,(ii)}$ $\underline{conv}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\ln n}{n^{0.1}}\right)^2$$

$$= \left(\lim_{n\to\infty} \frac{\ln n}{n^{0.1}}\right)^2 \underset{\ell'\text{Hôpital}}{=} \left(\lim_{n\to\infty} \frac{\frac{1}{n}}{0.1\,n^{-0.9}}\right)^2$$

$$= 0$$

Eg 8 $\sum a_n$ $\sum b_n$ Thm   Result

(g) $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ II (i)   conv

$$\because \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = \lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right)\left(\frac{1}{\cos\theta}\right) = 1$$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ II (i)   div

$$\therefore \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} n^{\frac{1}{n}} = 1.$$

$$\sum a_n \qquad \sum b_n \qquad \text{Thm} \qquad \text{Res.}$$

(2) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{\ln n}{n}} \qquad \sum \frac{1}{\sqrt{n}} \qquad 10 \qquad$ div.

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,\ln n} \qquad \sum_{n=1}^{\infty} \frac{1}{n^p} \quad 11 \text{ (iii)}$ div.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (c = +\infty)$

$$P = \tfrac{1}{2} \longrightarrow \text{inconclusive}.$$

$$P = 1, \tfrac{3}{4}, 0.9, 0.51, \dots$$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{1}{\sqrt{n}\,\ln n}}{\frac{1}{n}}$$

$$(\text{take } p=1) = \lim_{n \to \infty} \left(\frac{n}{\ln n}\right)^{\frac{1}{2}} = \infty$$

$$\text{Since } \sum b_n = \sum \tfrac{1}{n} = \infty$$
$$\Rightarrow \sum a_n = \infty$$