The Integral Test

<u>Thm</u> If $a_n > 0$, $a_n = f(n)$
where $f(x)$ is positive,
continuous and decreasing
for all $x \geq N$, $N \in \mathbb{N}$ $\Bigg\} (*)$

Then $\sum_{n=N}^{\infty} a_n < \infty \Longleftrightarrow \int_N^{\infty} f(x)\,dx < \infty$

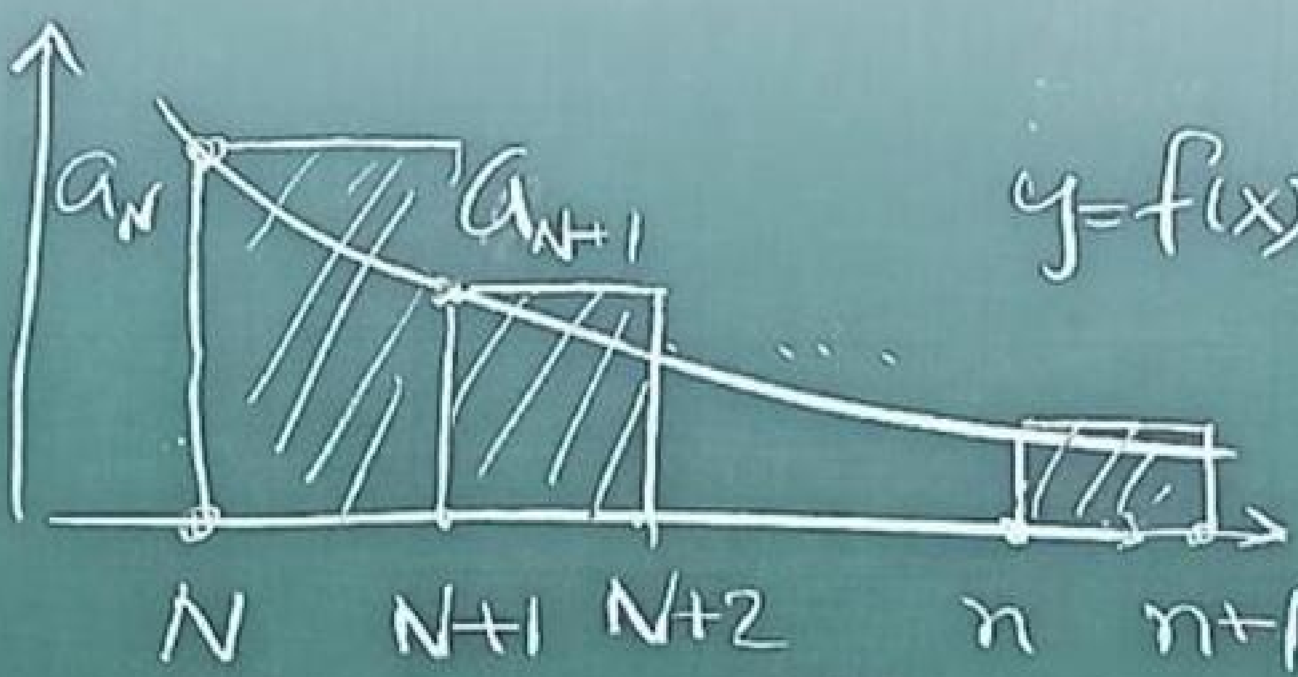
$\left(a_n > 0 \ \therefore \sum_{n=N}^{\infty} a_n < \infty \Leftrightarrow \sum_{n=N}^{\infty} a_n \text{ conv.} \right)$

Eg: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges?

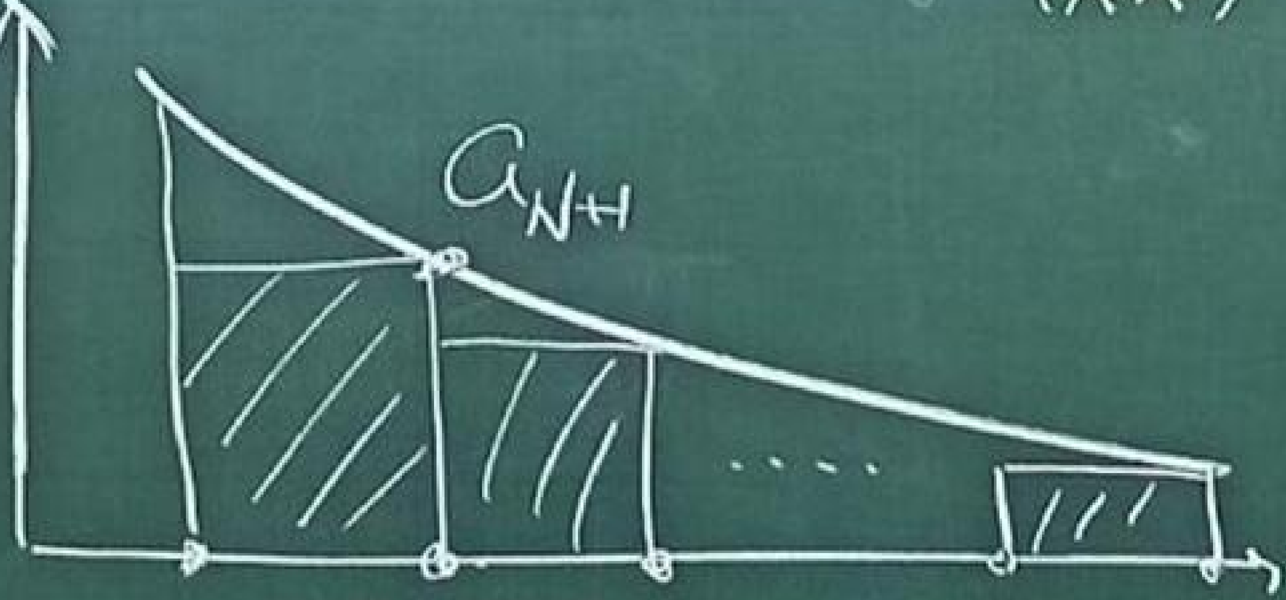
<u>Sol</u> $p \leq 0 \Rightarrow \lim_{n\to\infty} \frac{1}{n^p} \neq 0 \ \therefore \sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$

$p > 0 \Rightarrow$ Compare with $\int_1^{\infty} \frac{1}{x^p}\,dx \begin{cases} < \infty & p > 1 \\ = \infty & 0 < p \leq 1 \end{cases}$

Pf:



$$\Rightarrow \sum_{n=N}^{\infty} a_n \geq \int_N^{\infty} f(x)\,dx \quad (**)$$



$$\sum_{n=N+1}^{\infty} a_n \leq \int_N^{\infty} f(x)\,dx \quad (***)$$

$$\int_N^\infty f(x)dx \le \sum_{n=N}^\infty a_n \le a_N + \int_N^\infty f(x)dx \qquad (**)$$

(a_N is a fixed number)

$$\therefore \sum_{n=N}^\infty f(x)dx < \infty \Leftrightarrow \sum_{n=N}^\infty a_n < \infty$$

$\underline{R_M}$. If (*) holds, and

$\sum_{n=N}^\infty a_n = L < \infty$, then

we can estimate $(M > N)$

$$R_M = \sum_{n=N}^\infty a_n - \sum_{n=N}^M a_n \left(= \sum_{n=M+1}^\infty a_n\right)$$

$$\int_{M+1}^\infty f(x)dx \le \underbrace{\sum_{n=M+1}^\infty a_n}_{=R_M} \le \int_M^\infty f(x)dx$$

($N \to M+1$ in (**)) ($N \to M$ in (***)

Eg1. $\displaystyle\sum_{n=1}^{\infty} n^{-p} \begin{cases} =\infty, & 0<p\le 1 \\ <\infty, & p>1 \end{cases}$

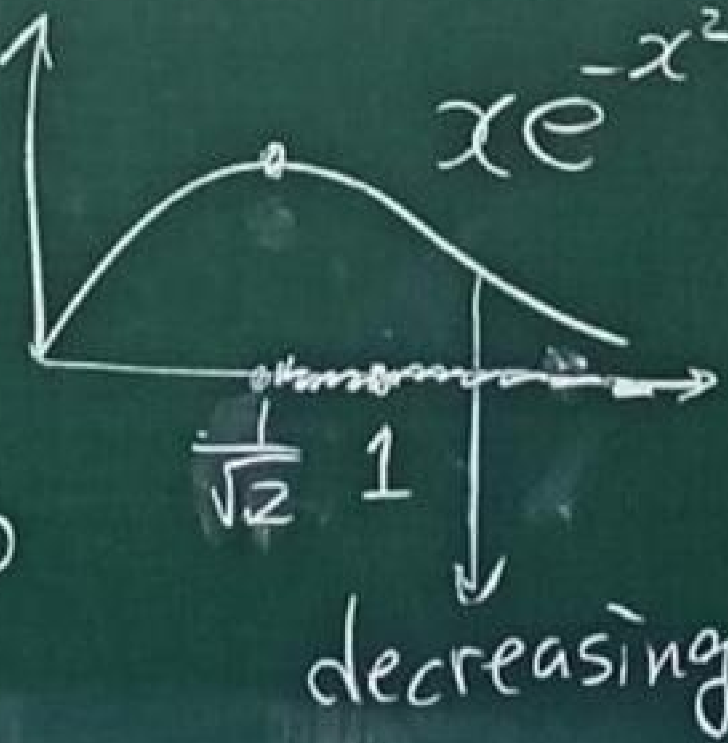
Eg2. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+n^2} < \infty$

<u>Sol</u>: $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_0^{b} \frac{1}{1+x^2}\,dx$

$\displaystyle = \lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 0 = \frac{\pi}{2} < \infty$

Eg3. $\displaystyle\sum_{n=1}^{\infty} n e^{-n^2} < \infty$

<u>Sol</u> $\displaystyle\int_1^{\infty} x e^{-x^2}\,dx$



$\displaystyle = -\frac{1}{2} e^{-x^2} \Big|_1^{\infty} < \infty$

$\displaystyle \Rightarrow \sum_{n=1}^{\infty} n e^{-n^2} < \infty$

Eg4 $\sum_{n=2}^{\infty} \frac{1}{2^{\ln n}} \leqq \infty$?

<u>Sol</u>: $\int_{2}^{\infty} \frac{1}{2^{\ln x}} dx = ?$

$y = \ln x$, $x = e^{y}$, $dx = e^{y} dy$

$= \int_{y=\ln 2}^{\infty} 2^{-y} e^{y} dy$

$= \int_{\ln 2}^{\infty} \left(\frac{e}{2}\right)^{y} dy = \infty$

$\frac{e}{2} > 1$

$\therefore \sum_{n=2}^{\infty} \frac{1}{2^{\ln n}} = \infty$

Direct Comparison Test

If $0 < d_n \le a_n \le c_n$

for all $n > N$, then

① $\sum_{n=1}^{\infty} c_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$

② $\sum_{n=1}^{\infty} d_n = \infty \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$

Eg 5 $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

Sol $\frac{5}{5n-1} > \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$\Rightarrow \sum_{n=1}^{\infty} \frac{5}{5n-1} = \infty$

Limit Comparison Test $\left(\sum_{n=1}^{\infty} a_n \overset{?}{<} \infty\right)$

If $a_n > 0$, $b_n > 0$ $\forall n \geq N$

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$, $0 < C < \infty$

Then $\sum_{n=1}^{\infty} b_n \underset{(=\infty)}{<\infty} \Leftrightarrow \sum_{n=1}^{\infty} a_n \underset{(=\infty)}{<\infty}$

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

Then $\sum_{n=1}^{\infty} b_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = +\infty$

Then $\sum_{n=1}^{\infty} b_n = \infty \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$

Eg 6 $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n+1}$

Sol $\dfrac{5}{5n+1} < \dfrac{1}{n}$

$\therefore$ Direct Comparison Test is inconclusive,

Limit Comparison Test

$$\lim_{n\to\infty} \frac{\frac{5}{5n+1}}{\frac{1}{n}} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{5}{5n+1} = \infty$$

Eg 7: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} < \infty \quad (0! \equiv 1)$

Sol $\because 1 + \dfrac{1}{1} + \dfrac{1}{1\cdot 2} + \dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots$

$< 1 + \dfrac{1}{1} + \dfrac{1}{1\cdot 2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots < \infty$

Eg 8:	$\sum a_n$	$\sum b_n$	Thm	Result
ⓐ	$\displaystyle\sum_{n=1}^{\infty}\frac{2n}{(n+1)^2}$	$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$	11 (i) $(c=1)$	div.
ⓑ	$\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n-1}$	$\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$	11 (i) $(C=1)$	conv.
ⓒ	$\displaystyle\sum_{n=1}^{\infty}\frac{1+n\ln n}{n^2+5}$	$\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$	11 (i) $(C=1)$	div

$$\sum_{n=1}^{\infty}\frac{\ln n}{n} \begin{array}{l} < \infty \\ = \infty \end{array} ?$$

Thm 10: Direct Comparison

Thm 11: Limit Comparison

Method 1. $\displaystyle\sum\frac{\ln n}{n} > \sum\frac{1}{n} = \infty$

Method 2: $\displaystyle\int_3^{\infty}\frac{\ln x}{x}\,dx = \infty \Rightarrow \sum_{n=3}^{\infty}\frac{\ln n}{n} = \infty$

($\frac{\ln x}{x}$ is decreasing for $x > e = 2.718\ldots$)

$\sum a_n$	$\sum b_n$	Thm	Result
ⓓ $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$	$\sum_{n=1}^{\infty} \frac{1}{n}$	11 (i) $(c=1)$	div
ⓔ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$	$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$	11 (i) $(c=1)$	conv
ⓕ $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{\frac{3}{2}}}$	$\sum_{n=1}^{\infty} \frac{n^{0.2}}{n^{\frac{3}{2}}}$	11 (ii) $(c=0)$	conv

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\ln n}{n^{0.1}}\right)^2$$

$$= \left(\lim_{n\to\infty} \frac{\ln n}{n^{0.1}}\right)^2 \underset{\text{L'Hôpital}}{=} \left(\lim_{n\to\infty} \frac{\frac{1}{n}}{0.1 n^{-0.9}}\right)^2$$

$$= 0$$

Eg 8 $\sum a_n$ $\sum b_n$ Thm Result

(g) $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n^2}\right)$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\overset{||(i)}{(c=1)}$ conv

$\because \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right)\left(\frac{1}{\cos\theta}\right) = 1$

(h) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ $\sum_{n=1}^{\infty} \frac{1}{n}$ $\overset{||(i)}{(c=1)}$ div

$\because \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} n^{\frac{1}{n}} = 1.$

$\sum a_n$ $\sum b_n$ Thm Res

(i) $\sum_{n=1}^{\infty} \sqrt{\frac{\ln n}{n}}$ $\sum \frac{1}{\sqrt{n}}$ 10 div.

(j) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n \ln n}}$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ 11(iii) div.
$(c=+\infty)$

$p=\frac{1}{2} \longrightarrow$ inconclusive.

$p=1, \frac{3}{4}, 0.9, 0.51, \cdots$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n \ln n}}}{\frac{1}{n}}$$

(take $p=1$) $= \lim_{n\to\infty} \left(\frac{n}{\ln n}\right)^{\frac{1}{2}} = \infty$

Since $\sum b_n = \sum \frac{1}{n} = \infty$

$\Rightarrow \sum a_n = \infty$