

"Sequence" = $\{a_0, a_1, a_2, \dots\}$
(= $\{a_n\}_{n=0}^{\infty}$)

How to define

$$\lim_{n \rightarrow \infty} a_n = L \text{ ?}$$

Recall the definition of

$$\lim_{x \rightarrow \infty} f(x) = L$$

and Replace x by n
 $f(x)$ by a_n

Def: $\lim_{n \rightarrow \infty} a_n = L$

if for every $\varepsilon > 0$, ($\varepsilon \in \mathbb{R}$)

there exists a corresponding integer N , such that

$$"n > N \Rightarrow |a_n - L| < \varepsilon"$$

(for all $n > N$, we have
 $|a_n - L| < \varepsilon$)

Def. $\lim_{n \rightarrow \infty} a_n = \pm \infty$

if for every $M \in \mathbb{R}$
($m \in \mathbb{R}$)

there exists a corresponding
integer N , such that

" $n > N \implies a_n > M$ "
($a_n < m$)

(for all $n > N$,
we have $a_n > M$
($a_n < m$))

$$\text{Eg 1. } \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\text{Eg 2. } \lim_{n \rightarrow \infty} n^2 = +\infty$$

$$\text{Eg 3 } \{a_n\} = \{1, -1, 1, -1, \dots\}$$

$$(a_n = (-1)^{n-1}), \lim_{n \rightarrow \infty} a_n \text{ diverges}$$

$$\text{Eg 4 } \{a_n\} = \{1, 0, 2, 0, 3, 0, \dots\}$$

$$\lim_{n \rightarrow \infty} a_n \text{ diverges.}$$

How to find $\lim_{n \rightarrow \infty} a_n$?

(I): Apply Sandwich Thm
for sequences (Section 10.1
Theorem 2)

Ex 5: $\lim_{n \rightarrow \infty} \frac{\cos n}{n} = ?$

Sol: $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$

$$\lim_{n \rightarrow \infty} \frac{-1}{n} = 0 = \lim_{n \rightarrow \infty} \frac{1}{n}$$

Thm 2 $\implies \lim_{n \rightarrow \infty} \frac{\cos n}{n} = 0$

(II) Thm 4 Suppose

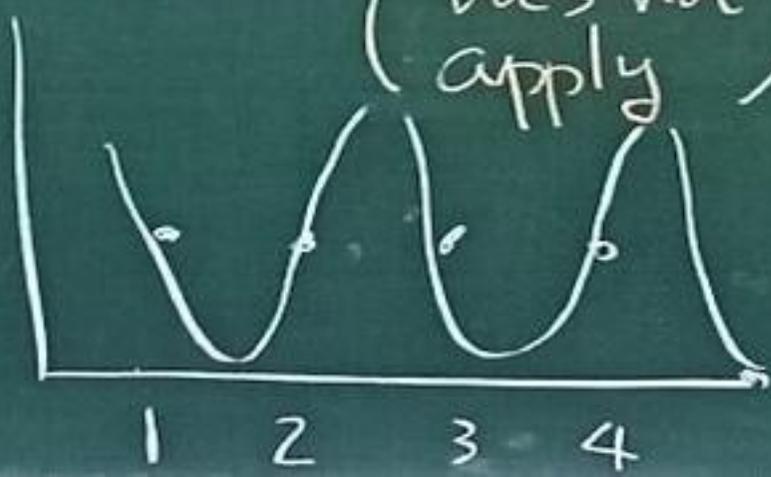
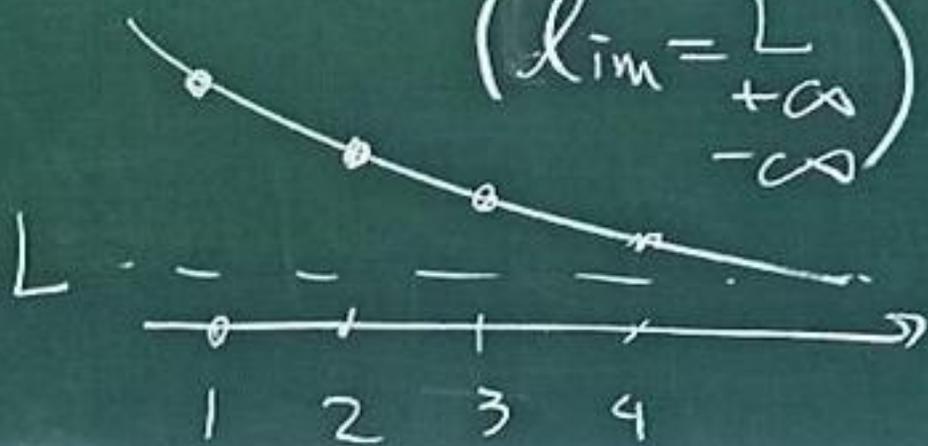
$f(x)$ is a function defined
for all $x \geq n_0$ with $a_n = f(n)$

and $\lim_{x \rightarrow \infty} f(x) = \begin{matrix} L \\ +\infty \\ -\infty \end{matrix}$

then $\lim_{n \rightarrow \infty} a_n = \lim_{x \rightarrow \infty} f(x)$

($\lim = \begin{matrix} L \\ +\infty \\ -\infty \end{matrix}$)

(Does not apply)



$$\text{Ex 6 } \lim_{n \rightarrow \infty} \frac{\ln n}{n} = ?$$

Ans. Use Thm 4.

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} \left(= \frac{\infty}{\infty} \right)$$

$(x \in \mathbb{R}, x > 0)$

L'Hôpital Rule

$$= \lim_{x \rightarrow \infty} \frac{(\ln x)'}{x'}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{x} = 0 \therefore \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

$$\text{Ex 7 } \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = ?$$

$$= \lim_{n \rightarrow \infty} (e^{\ln n})^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} e^{\frac{\ln n}{n}}$$

$$\stackrel{\text{Thm 3}}{=} e^{\left(\lim_{n \rightarrow \infty} \frac{\ln n}{n} \right)}$$

$$= e^0 = 1$$

Since $x \xrightarrow{f} e^x$

is continuous

$$\therefore \lim_{x \rightarrow x_0} f(x) = f\left(\lim_{x \rightarrow x_0} x\right)$$

$$\text{Ex 8. } \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = ?$$

Here $x \in \mathbb{R}$ is fixed

Sol.: " 1^∞ "

$$= \lim_{n \rightarrow \infty} \left(e^{\ln\left(1 + \frac{x}{n}\right)} \right)^n$$

$$= \lim_{n \rightarrow \infty} e^{\frac{\ln\left(1 + \frac{x}{n}\right)}{n}}$$

$$= e^{\lim_{n \rightarrow \infty} \left(\frac{\ln\left(1 + \frac{x}{n}\right)}{n} \right)}$$

$$\lim_{n \rightarrow \infty} \frac{\ln\left(1 + \frac{x}{n}\right)}{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{d}{dn}\left(\ln\left(1 + \frac{x}{n}\right)\right)}{\frac{d}{dn}\left(\frac{1}{n}\right)}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{1}{1 + \frac{x}{n}} \left(-x n^{-2}\right)}{-n^{-2}}$$

$$= x$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

$$\text{Ex 9} \quad \lim_{n \rightarrow \infty} \left(\frac{n+1}{n-1} \right)^n$$

$$\underline{\text{Sol}} = \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n-1} \right)^n$$

$$= \lim_{n \rightarrow \infty} \left(\left(1 + \frac{2}{n-1} \right)^{n-1} \cdot \left(1 + \frac{2}{n-1} \right) \right)$$

$$= \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n-1} \right)^{n-1} \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n-1} \right)$$

$$= e^2 \cdot 1 = e^2$$

Infinite Series

Def. $\sum_{n=1}^{\infty} a_n = L$
($\pm\infty$)

if $\lim_{n \rightarrow \infty} S_n = L$
($\pm\infty$)

where $S_n = \sum_{k=1}^n a_k$

($= a_1 + a_2 + \dots + a_n$)

i.e. $S_1, S_2, \dots, S_n,$
is the sequence of
partial sum of $\{a_k\}$

Examples of convergent Series

* Geometric Series

$$\sum_{n=0}^{\infty} ar^n = a + ar + \dots + ar^n + \dots$$

$$S_n = \sum_{k=0}^n ar^k = a \frac{1-r^{n+1}}{1-r}$$

$$\sum_{n=0}^{\infty} ar^n = \lim_{n \rightarrow \infty} a \frac{1-r^{n+1}}{1-r}$$

$$= \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{div.} & \text{if } |r| \geq 1 \end{cases}$$

* Telescoping Sum

Eg 10 $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Sol $S_n = \sum_{k=1}^n \frac{1}{k(k+1)}$

$$= \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{2} \right)$$

$$+ \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$+ \dots$$

$$+ \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= 1 - \frac{1}{n+1} \quad \therefore \sum_{n=1}^{\infty} = \lim_{n \rightarrow \infty} S_n = 1$$

Thm (Section 10.2, Thm 7)

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \implies \lim_{n \rightarrow \infty} a_n = 0$$

$$\left(\sum_{n=1}^{\infty} a_n \text{ div} \iff \lim_{n \rightarrow \infty} a_n \neq 0 \text{ or does not exist} \right)$$

The n-th term test

pf: Since $a_n = S_n - S_{n-1}$

$$\lim_{n \rightarrow \infty} a_n = \left(\lim_{n \rightarrow \infty} S_n \right) - \left(\lim_{n \rightarrow \infty} S_{n-1} \right)$$

Eg of ~~" \iff "~~ $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Thm (Section 10.2, Thm 7)

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \implies \lim_{n \rightarrow \infty} a_n = 0$$

~~(*)~~

$$\left(\sum_{n=1}^{\infty} a_n \text{ div} \iff \lim_{n \rightarrow \infty} a_n \neq 0 \text{ or does not exist} \right)$$

pf: Since $a_n = S_n - S_{n-1}$

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Eg of ~~(*)~~ $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Examples of divergent Series

Eg 11 ($\lim_{n \rightarrow \infty} a_n \neq 0$ or D.N.E.)

$$\sum_{n=1}^{\infty} n^2, \quad \sum_{n=1}^{\infty} (-1)^n, \quad \sum_{n=1}^{\infty} \frac{n}{2n+5}$$

all diverge

Eg 12 $\sum_{n=1}^{\infty} \frac{1}{n}$

Sol: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$

$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

$$= +\infty$$