Thm 2. If f, g are cont. on $[a, \infty)$ and $0 \le f \le g$

Then

(1) $\int_a^\infty g(x)\,dx < \infty \Rightarrow \int_a^\infty f(x)\,dx < \infty$

(conv.) (conv)

(2) $\int_a^\infty f(x)\,dx = \infty \Rightarrow \int_a^\infty g(x)\,dx = \infty$

Similarly for $\lim_{x \to -\infty}$ and type II.

Eg 1 $\int_0^\infty e^{-x^2}\,dx$ conv?

Sol Compare with $\int_0^\infty e^{-x}\,dx$

$$e^{-x^2} \leq e^{-x} \text{ on } x \geq 1$$

$$\int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty}\left(e^{-1} - e^{-b}\right) < \infty$$

$$\Rightarrow \int_1^\infty e^{-x^2}\,dx < \infty \Rightarrow \int_0^\infty e^{-x^2}\,dx < \infty$$

Thm 3 If f, g are cont. on $[a, \infty)$ and $f>0$, $g>0$.

and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$

(1) $0 < L < \infty$

$$\int_a^\infty f(x)dx < \infty \Leftrightarrow \int_a^\infty g(x)dx < \infty$$

(2) $L=0$ " $\Leftarrow$ "

(3) $L=\infty$ " $\Rightarrow$ "

Similarly for $(-\infty, b]$ and type II

Eg 2. Check convergence

	$\int_{*}^{*} f(x)\,dx$	$g(x)$	Thm	conv/div
ⓐ	$\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$	$\frac{1}{x^2}$	2-(1)	conv
ⓑ	$\int_1^{\infty} \frac{1}{\sqrt{x^2+1}}\,dx$	$\frac{1}{x}$	3-(1)	div.
ⓒ	$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{x}}\,dx$	$\frac{1}{\sqrt{x}}$	$\begin{cases} 2\text{-}(1) \\ 3\text{-}(1) \\ x \to 0^+ \end{cases}$	conv
ⓓ	$\int_1^{\infty} \frac{1-e^{-x}}{x}\,dx$	$\frac{1}{x}$	3-(1)	div.

Eg 3. $\int_1^\infty \frac{1}{x(x^2-1)^{\frac{1}{3}}}dx$ conv?

$\underline{\text{Sol}}$ $= \int_1^2 + \int_2^\infty$

$$f(x) = \frac{1}{x(x^2-1)^{1/3}}$$

(a) $\int_1^2$: take $g(x) = \frac{1}{(x-1)^{\frac{1}{3}}}$

$$\lim_{x\to 1^+} \frac{f(x)}{g(x)} = 2^{\frac{-1}{3}}$$

$$\int_1^2 g(x)\,dx < \infty \underset{(p=\frac{1}{3})}{} \overset{\text{Thm 3-(1)}}{\Longrightarrow} \int_1^2 f(x)\,dx < \infty$$

ⓑ $\int_2^\infty$: take $g(x)=\dfrac{1}{x^{1+\frac{2}{3}}}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{(x^2)^{\frac{1}{3}}}{(x^2-1)^{\frac{1}{3}}}=1$$

$$\int_2^\infty g(x)\,dx<\infty \quad \left(p=\frac{5}{3}\right)$$

$$\text{Thm } 3\text{-}(1) \Rightarrow \int_2^\infty f(x)\,dx<\infty$$

$$ⓐ, ⓑ \Rightarrow \int_1^\infty f(x)\,dx<\infty$$

Eg4 $\int_e^\infty \underbrace{x^{-2}\ln x}_{f(x)}\,dx$ conv?

Take $g(x) = x^{-2} \longrightarrow$ No conclusion

Take $g(x) = x^{-1.5}$ instead

($x^{-1.9}, x^{-1.99} \ldots$ etc)

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\ln x}{x^{0.5}}$$

$=$ (l'Hôpital) $= 0$

Thm 3-(2)

$\Rightarrow \int_e^\infty f(x) \ln x\,dx < \infty$

Eg5 $\int_e^\infty x^{-0.5} \ln x \, dx$

Sol take $g(x) = x^{-0.5}$

$f(x) > g(x)$, $\int_e^\infty g(x)\,dx = \infty$

$\Rightarrow \int_e^\infty f(x)\,dx = \infty$

Eg6. $\int_0^{1/e} x^{-0.5}(-\ln x)\,dx$

Sol take $g(x) = x^{-\frac{2}{3}}$, $\int_0^{\frac{1}{e}} x^{-\frac{2}{3}}\,dx < \infty$

$\lim_{x \to 0^+} \frac{x^{-0.5}(-\ln x)}{x^{-\frac{2}{3}}} = \lim_{x \to 0^+} \left(-\frac{\ln x}{x^{-\frac{1}{6}}}\right) \overset{\text{l'Hôpital}}{=} 0$

Thm 3-(2) $\Rightarrow \int_0^{\frac{1}{e}} f(x)\,dx$ conv.

Eg 5 $\int_e^\infty x^{-0.5} \ln x \, dx$

Sol take $g(x) = x^{-0.5}$

$f(x) > g(x)$, $\int_e^\infty g(x)\,dx = \infty$

$\Rightarrow \int_e^\infty f(x)\,dx = \infty$

Eg 6. $\int_0^{1/e} \underline{x^{-0.5}(-\ln x)}\,dx$ $> x^{-0.5}$

Inconclusive if we take $g(x) = x^{-0.5}$

Sol take $g(x) = x^{-\frac{2}{3}}$, $\int_0^{\frac{1}{e}} x^{-\frac{2}{3}}\,dx < \infty$

$\lim_{x\to 0^+} \frac{x^{-0.5}(-\ln x)}{x^{-\frac{2}{3}}} = \lim_{x\to 0^+}\left(-\frac{\ln x}{x^{-\frac{1}{6}}}\right) \overset{\text{l'Hôpital}}{=} 0$

Thm 3-(2) $\Rightarrow \int_0^{\frac{1}{e}} f(x)\,dx$ conv.