

Improper Integrals (瑕積分)

Recall (proper) integrals:

If $f(x)$ is cont. on $[a, b]$

$$\implies |f(x)| \leq M \text{ on } [a, b]$$

$$\implies \int_a^b f(x) dx \text{ exists}$$

If either $(b-a)$ or $\max|f|$
becomes $\infty \implies$ Improper integrals

Def: (Type I)

(1) If f is cont. on $[a, \infty)$

$$\int_a^{\infty} f(x) dx \stackrel{\text{def}}{=} \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

(2) If f is cont. on $(-\infty, b]$

$$\int_{-\infty}^b f(x) dx \stackrel{\text{def}}{=} \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

(3) If f is cont. on $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} f(x) dx \stackrel{\text{def}}{=} \int_{-\infty}^c + \int_c^{\infty} \quad \text{for any } c \in \mathbb{R}$$

Limit exists \iff Improper
and finite integral converges
Otherwise \iff diverges

In case (3), $\int_{-\infty}^{\infty} f(x) dx$

Converges if both
 $\int_{-\infty}^c$ and \int_c^{∞} exist and
are finite (converge)

If either $\int_{-\infty}^c$ or \int_c^{∞} diverges
 $\iff \int_{-\infty}^{\infty}$ diverges

Remark: If $f \geq 0$
then \int_a^∞ , $\int_{-\infty}^b$ or $\int_{-\infty}^\infty$
either converges or
diverges to $+\infty$.

We write

$$\int_a^\infty f(x) dx = \infty$$

to mean it diverges
to $+\infty$

$$\text{Eg 1 } \int_1^{\infty} \frac{\ln x}{x^2} dx = ?$$

$$\text{Sol: } \int_1^b \frac{\ln x}{x^2} dx = \int_1^b \ln x d\left(\frac{-1}{x}\right)$$

$$= (\ln x) \left(\frac{-1}{x}\right) \Big|_1^b - \int_1^b \frac{-1}{x} d \ln x$$

$$= \frac{-1}{b} \ln b + \int_1^b \frac{1}{x} \left(\frac{dx}{x}\right)$$

$$= \frac{-1}{b} \ln b - \left(\frac{1}{b} - 1\right)$$

$$\int_1^{\infty} = \lim_{b \rightarrow +\infty} \int_1^b = 1$$

Ex 2: $\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx$ converges?

Sol: Can we do

$\lim_{b \rightarrow \infty} \int_{-b}^b \frac{x}{1+x^2} dx$? No!

($= \lim_{b \rightarrow \infty} 0 = 0$)

Correct computation:

Find \int_0^{∞} and $\int_{-\infty}^0$ separately.

$$\begin{aligned} \lim_{b \rightarrow \infty} \int_0^b \frac{x}{1+x^2} dx &= \lim_{b \rightarrow \infty} \frac{1}{2} \int_0^b \frac{d(x^2)}{1+x^2} \\ &= \lim_{b \rightarrow \infty} \frac{1}{2} \int_0^b \frac{d(1+x^2)}{1+x^2} = \lim_{b \rightarrow \infty} \frac{\ln(1+b^2)}{2} = \infty \end{aligned}$$

Since \int_0^{∞} diverges

$\Rightarrow \int_{-\infty}^{\infty}$ diverges (by definition)

Remark:

$$\lim_{b \rightarrow \infty} \int_{-b}^{2b} \frac{x}{1+x^2} dx$$

$$= \lim_{b \rightarrow \infty} \frac{1}{2} \ln \left(\frac{1+(2b)^2}{1+b^2} \right) \quad \leftarrow$$

$$= \ln 2$$

This example explains why we must define $\int_{-\infty}^{\infty}$ as $\int_{-\infty}^c + \int_c^{\infty}$

Ex 3 $\int_1^{\infty} \frac{1}{x^p} dx, p > 0$

Ans $\begin{cases} p > 1 & \text{converges} \\ 0 < p \leq 1 & \text{diverges} \end{cases}$ (21t)
A

Sol

$$p \neq 1 \quad \int_1^{\infty} x^{-p} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-p} dx$$

$$= \lim_{b \rightarrow \infty} \left(\frac{1}{-p+1} x^{-p+1} \Big|_1^b \right)$$

$$p=1 \quad \int_1^{\infty} x^{-1} dx = \lim_{b \rightarrow \infty} (\ln x \Big|_1^b) = \infty$$

Def. (type II)

(1) If $f(x)$ is cont. on $(a, b]$

$$\int_a^b f(x) dx \stackrel{\text{def}}{=} \lim_{c \rightarrow a^+} \int_c^b f(x) dx$$

(2) If $f(x)$ is cont. on $[a, b)$

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$$

(3) If $f(x)$ is cont. on (a, b)

$$\int_a^b f(x) dx \stackrel{\text{def}}{=} \int_a^c f(x) dx + \int_c^b f(x) dx$$

for any $c \in (a, b)$ (any $c \in (a, b)$ will do)

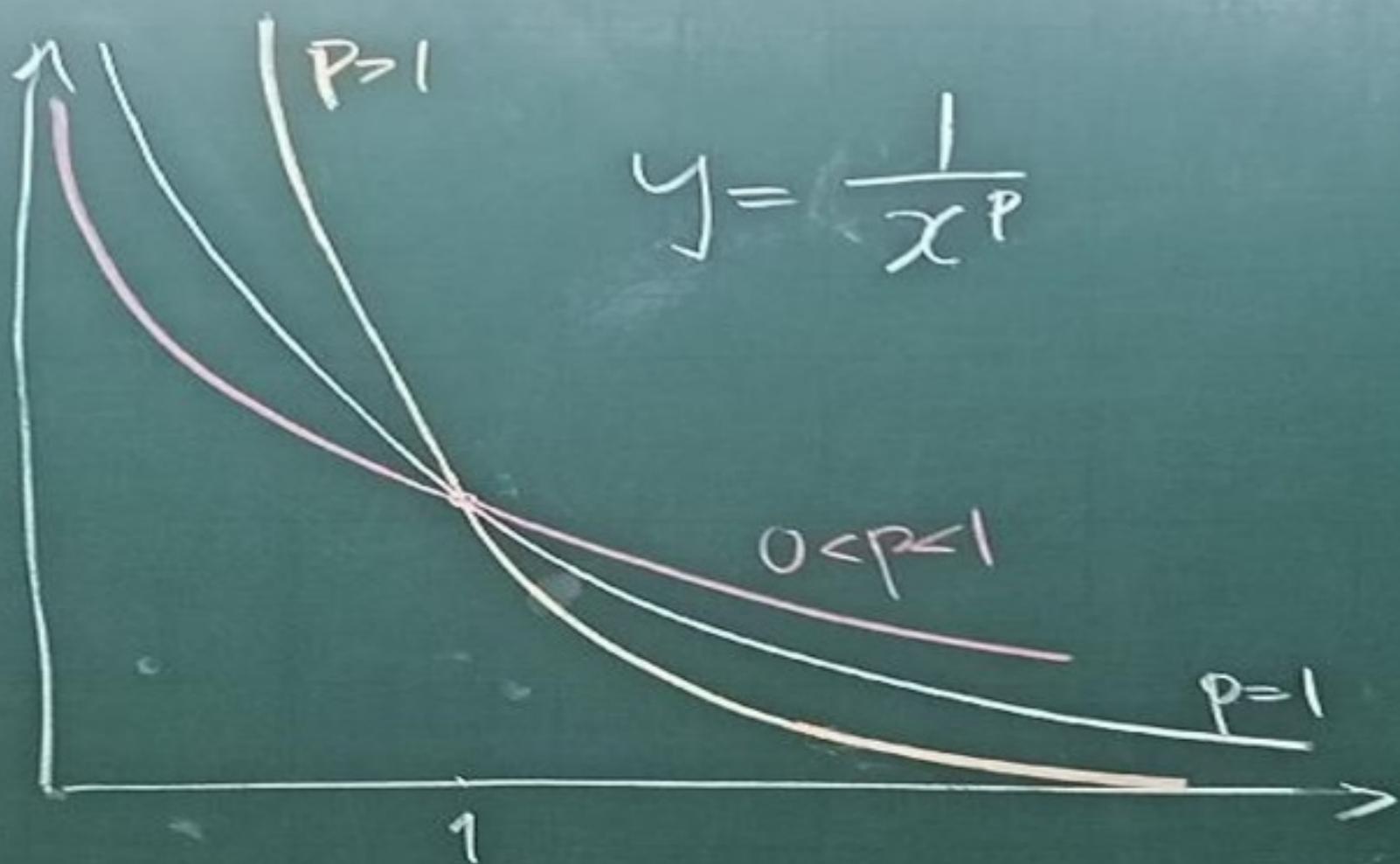
$$\text{Ex 4: } \int_0^1 \frac{1}{x^p} dx, \quad p > 0$$

Ans: $p \geq 1$: diverges
 $0 < p < 1$: converges ($\frac{1}{A}$)

$$p \neq 1: \int_0^1 x^{-p} dx = \lim_{c \rightarrow 0^+} \int_c^1 x^{-p} dx$$

$$= \lim_{c \rightarrow 0^+} \left(\frac{1}{-p+1} x^{-p+1} \Big|_c^1 \right)$$

$$p=1: \int_0^1 x^{-1} dx = \lim_{c \rightarrow 0^+} \left(\ln x \Big|_c^1 \right) \\ = +\infty$$



Remark: The same conclusion applies to $\int_c^{c+1} \frac{1}{(x-c)^p} dx$ and $\left(\int_{c-1}^c \frac{1}{(x-c)^p} dx \right)$ by the change of variable. $y = x-c$ ($c-x$)

Remark: Same conclusion

holds for $\int_2^{\infty} \frac{1}{x^p} dx$
or $\int_{-\infty}^{-1} \frac{1}{|x|^p} dx$
or $\int_{c+1}^{\infty} \frac{1}{(x-c)^p} dx$, etc

} Type I

and $\int_0^2 \frac{1}{x^p} dx$,
or $\int_{-1}^0 \frac{1}{|x|^p} dx$, etc

} Type II

Ex 5. $\int_0^1 \frac{1}{1-x} dx$ converge?

Ans: This is $p=1$ (type II)
it diverges

$$\text{Sol} = \lim_{c \rightarrow 1^-} \int_0^c \frac{1}{1-x} dx$$

let $y = 1-x$, $dx = -dy$

$$= \lim_{c \rightarrow 1^-} \int_{y=1}^{1-c} \frac{dy}{y} = \lim_{c \rightarrow 1^-} -\ln y \Big|_1^{1-c}$$

$$= \lim_{c \rightarrow 1^-} -\ln(1-c) = +\infty \text{ (diverges)}$$

Ex 6: $\int_0^{\infty} \frac{1}{x^p} dx$ converges?
($p > 0$)

Ans: Type I at $[1, \infty)$
+ Type II at $(0, 1]$

$$\int_0^{\infty} = \int_0^1 + \int_1^{\infty}$$

$$= \begin{cases} \text{finite} & + \infty & 0 < p < 1 \\ + \infty & + \infty & p = 1 \\ + \infty & + \text{finite} & p > 1 \end{cases}$$

= divergent (at least one of \int_0^1 or \int_1^{∞} diverges)

Ex 7. $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$ converges?

Sol: Check \int_0^1 and \int_1^3
separately. (both type II)

From previous remark

both $\int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}}$ and $\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$

converge by the change

of variable. $y=1-x$ or $y=x-1$

and $p = \frac{2}{3}$ (type II) $\Rightarrow \int_0^3$ converges