Homework 08

1. Section 14.5: Let

$$
\begin{aligned}
f_1(x,y) &= \sqrt{x^2+y^2}^{\frac{1}{2}} = (x^2+y^2)^{\frac{1}{4}}, \\
f_2(x,y) &= 2x+3y+4+\sqrt{x^2+y^2}^{\frac{3}{2}} = 2x+3y+4+(x^2+y^2)^{\frac{3}{4}} \\
f_3(x,y) &= \begin{cases} \dfrac{x^3}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}.
\end{aligned}
$$

   (a) Are f_i continuous at $(0,0)$?

   (b) Do $\partial_x f_i$ and $\partial_y f_i$ exist at $(0,0)$?

   (c) Use the definition of directional derivative to evaluate $\left.\dfrac{df_i}{ds}\right|_{(0,0),(\cos\theta,\sin\theta)}$, i.e. the directional derivative of f_i at $(x_0,y_0)=(0,0)$ in the direction $(\cos\theta,\sin\theta)$, if it exists.

   (d) Are f_i differentiable at $(0,0)$?
   Hint:

      i. If you know or guess that f is differentiable at (x_0,y_0), you can try to prove it using Theorem 3 (Section 14.3, page 832). It may or may not work.

      ii. If you know or guess that f is NOT differentiable at (x_0,y_0), you can try to prove it using Theorem 4 (Section 14.3, page 832) or Theorem 9 (Section 14.5, page 847). It may or may not work.

      iii. Note: You definitely CANNOT use Theorem 3 in (ii) or Theorem 4, Theorem 9 in (i). If you don't know whether f is differentiable at (x_0,y_0) or not, it is always safe to follow the definition as outlined in Lecture 13. It almost always works. See also page 3 of Remark_on_definition_of_differentiability_v03.pdf.

2. Section 14.6: Problems 5, 9, 17, 19, 25(b), 33, 39(a), 45, 55, 57, 58.

   Remark: Suppose that $f(x,y)$ is differentiable at (x_0,y_0). Both "linearization" and "linear approximation" of $f(x,y)$ at (x_0,y_0) refer to the tangent plane of the surface $z=f(x,y)$ at $(x_0,y_0,f(x_0,y_0))$. That is, the plane $z=L(x,y)$ given on page 856.

   Problem 55 provides another way of finding the tangent plane $z=L(x,y)$. That is, apply the definition and equation (1) on page 853 to the function $F(x,y,z)=z-f(x,y)$:

$$\nabla F(x_0,y_0,z_0)\cdot(x-x_0,y-y_0,z-z_0)=0$$

   Verify that this will lead to the equation $z=L(x,y)$.

   Hint for problem 39, 45: See page page 859 (item 1 and 2) for generalization to functions of 3 variables.

3. Section 14.7: Problems 1, 19, 31, 35, 39, 43, 44, 49, 51.

   Alternative method for problems 31, 35: To find absolute minimum and absolute maximum, find all local minima and local maxima first, by plotting the gradient vectors inside the domain and tangential component of the gradient vectors on the boundary as done in the Lecture. Then compare all the local minima and local maxima to find the absolute ones.

   Hint for problem 44: Do this problem using the gradient analysis. That is, plot ∇f near the critical point to determine whether the critical point is a local minimum, local maximum or neither.

   In some cases, there may be other methods other than plotting ∇f, but plotting ∇f is more systematic and guaranteed to work.

   Hint for problem 49, 51: Minimizing/maximizing distance is the same as minimizing/maximizing (distance)^2. The latter is easier to compute.