

Homework 07

1. Section 14.3 (Part II): Problems 72, 91.

Hint for problem 72:

To compute

$$f_{xy}(0, 0) = \lim_{y \rightarrow 0} \frac{f_x(0, y) - f_x(0, 0)}{y - 0},$$

we need $f_x(0, y)$ for $y \neq 0$ (direct differentiation) and

$$f_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0}.$$

The procedure for computing $f_{yx}(0, 0)$ is similar.

2. Section 14.3: (Equivalent definitions of differentiability on Lecture 13, page 7)

Show that

$$h(x, y) = \varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0) \text{ with } \lim_{(x,y) \rightarrow (x_0,y_0)} (\varepsilon_1, \varepsilon_2) = (0, 0) \quad (1)$$

is the same as

$$h(x, y) = \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2} \text{ with } \lim_{(x,y) \rightarrow (x_0,y_0)} \varepsilon = 0. \quad (2)$$

That is, if $h(x, y)$ is given by (1), then it can be rewritten (combined) as (2). Also, if $h(x, y)$ is given by (2), then it can be rewritten (split) as (1).**Hint:**

$$(1) \implies (2): \text{ Use } |\Delta x| \leq \sqrt{(\Delta x)^2 + (\Delta y)^2}, \quad |\Delta y| \leq \sqrt{(\Delta x)^2 + (\Delta y)^2},$$

$$(2) \implies (1): \text{ Use } \sqrt{(\Delta x)^2 + (\Delta y)^2} \leq |\Delta x| + |\Delta y|$$

or

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \Delta x + \frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \Delta y$$

Remark: In terms of the notation ‘ O ’ (‘Big O’) and ‘ o ’ (‘Big o’) in Section 7.4 (page 464), the above statement can be recast as:

$$o(1) \cdot \Delta x + o(1) \cdot \Delta y = o(1) \cdot \sqrt{(\Delta x)^2 + (\Delta y)^2}, \quad (3)$$

or

$$o(\Delta x) + o(\Delta y) = o(\sqrt{(\Delta x)^2 + (\Delta y)^2}), \quad (4)$$

where the small $o(\cdot)$ ’s refer to $2D$ limit $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)}$, same as $\lim_{(x,y) \rightarrow (x_0,y_0)}$.

The statements (3) or (4) are more concise and widely used.

3. Section 14.4: Problems 1, 7, 10, 21, 24, 29, 31, 43, 51.

4. Section 14.5: Problems 9, 15, 19, 25, 27, 29, 35, 36.

5. Section 14.5: Read and memorize the Algebraic Rules for Gradients on page 850.