

## Brief solutions to selected problems in homework 13

### 1. Section 7.4: Solutions, common mistakes and corrections:

7.4.5

$$\begin{aligned} \text{a. } \lim_{x \rightarrow \infty} \frac{\log_3 x}{\ln x} &= \frac{1}{\ln 3} \Rightarrow \text{same} \\ \text{b. } \lim_{x \rightarrow \infty} \frac{\ln^2 x}{\ln x} &= \frac{1}{1} \Rightarrow \text{same} \\ \text{c. } \lim_{x \rightarrow \infty} \frac{\ln^2 x}{\ln x} &= \frac{1}{2} \Rightarrow \text{same} \\ \text{d. } \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln x} &= \infty \Rightarrow \text{faster} \\ \text{e. } \lim_{x \rightarrow \infty} \frac{x}{\ln x} &= \infty \Rightarrow \text{faster} \\ \text{f. } \lim_{x \rightarrow \infty} \frac{5 \ln x}{\ln x} &= 5 \Rightarrow \text{same} \end{aligned}$$

$$\begin{aligned} \frac{1/x}{\ln x} &= \frac{1}{x \ln x} \\ \text{g. } \lim_{x \rightarrow \infty} \left( \frac{x}{\ln x} \right) &= 0 \Rightarrow \text{slower} \\ \text{h. } \lim_{x \rightarrow \infty} \frac{e^x}{\ln x} &= \infty \Rightarrow \text{fast} \end{aligned}$$

Figure 1: Solution to Section 7.4, problem 5

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{e^{x/2}}{e^x} &= \lim_{x \rightarrow \infty} e^{-x/2} = 0 \\ \lim_{x \rightarrow \infty} \frac{(\ln x)^x}{e^x} &= \lim_{x \rightarrow \infty} \left( \frac{\ln x}{e} \right)^x = \infty \\ \lim_{x \rightarrow \infty} \frac{x^x}{(\ln x)^x} &= \lim_{x \rightarrow \infty} \left( \frac{x}{\ln x} \right)^x = \infty \\ \therefore \text{d, a, c, b} &\# \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x}{\ln x} \\ \text{L.H.} &= \lim_{x \rightarrow \infty} \frac{1}{1/x} = \infty \end{aligned}$$

Figure 2: Solution to Section 7.4, problem 7

$$n, \sqrt{n} \log_2 n, (\log_2 n)^2$$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n} \log_2 n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\log_2 n} = \infty$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n} \log_2 n}{(\log_2 n)^2} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\log_2 n} = \infty$$

rate growth  $n > \sqrt{n} \log_2 n > (\log_2 n)^2$

$$\Rightarrow (\log_2 n)^2$$

Figure 3: Solution to Section 7.4, problem 24

2. Section 8.2: Solutions, common mistakes and corrections:

$$\begin{aligned} \int e^{2x} \cos 3x dx &= \frac{1}{3} \sin 3x e^{2x} - \int \frac{1}{3} \sin 3x \cdot (e^{2x})' dx \\ &= \frac{1}{3} \sin 3x e^{2x} - \frac{2}{3} \int \sin 3x \cdot e^{2x} dx \\ &= \frac{1}{3} \sin 3x e^{2x} - \frac{2}{3} \left( \frac{1}{3} (-\cos 3x) e^{2x} - \int \frac{1}{3} (-\cos 3x) (e^{2x})' dx \right) \\ &= \frac{1}{3} \sin 3x e^{2x} + \frac{2}{9} \cos 3x e^{2x} - \frac{4}{9} \int \cos 3x e^{2x} dx \quad \text{移到左式} \\ \Rightarrow (1 + \frac{4}{9}) \int e^{2x} \cos 3x dx &= \frac{1}{3} \sin 3x e^{2x} + \frac{2}{9} \cos 3x e^{2x} \\ \Rightarrow \int e^{2x} \cos 3x dx &= \frac{3}{13} \sin 3x e^{2x} + \frac{2}{13} \cos 3x e^{2x} + C. \end{aligned}$$

Figure 4: Solution to Section 8.2, problem 23

$$\begin{aligned}
 & \int e^{\sqrt{3x+9}} dx \quad \text{let } u = \sqrt{3x+9} \\
 & dx = \frac{2}{3} \sqrt{3x+9} du \\
 & = \int e^u \frac{2}{3} \sqrt{3x+9} du \\
 & = \frac{2}{3} \int e^u u du \quad \begin{array}{l} f = u \\ df = du \end{array} \\
 & = \frac{2}{3} (u e^u - \int e^u du) \quad dg = e^u du \\
 & = \frac{2}{3} (e^u (u-1)) g = e^u \\
 & = \frac{2}{3} e^{\sqrt{3x+9}} (\sqrt{3x+9} - 1) + C
 \end{aligned}$$

Figure 5: Solution to Section 8.2, problem 25

$$\begin{aligned}
 & \int f dg = fg - \int g df \\
 & \int_0^{\frac{\pi}{3}} x \tan^2 x dx = \int_0^{\frac{\pi}{3}} x (\sec^2 x - 1) dx \\
 & = \int_0^{\frac{\pi}{3}} x d \tan x - \int_0^{\frac{\pi}{3}} x dx \\
 & = x \tan x \Big|_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan x dx - \frac{1}{2} x^2 \Big|_0^{\frac{\pi}{3}} \\
 & = \pi \frac{\sqrt{3}}{3} - \ln |\sec x| \Big|_0^{\frac{\pi}{3}} - \frac{\pi^2}{18} \\
 & = \pi \frac{\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18} \quad \checkmark
 \end{aligned}$$

Figure 6: Solution to Section 8.2, problem 27



$$\begin{aligned}
 & \int \sin(\ln x) dx \quad \left( \begin{array}{l} u = \sin(\ln x), \quad du = \frac{\cos(\ln x)}{x} dx \\ dv = dx, \quad v = x \end{array} \right) \\
 &= \sin(\ln x) \cdot x - \int \cos(\ln x) dx \\
 & \int \cos(\ln x) dx \quad \left( \begin{array}{l} u = \cos(\ln x), \quad du = \frac{-\sin(\ln x)}{x} dx \\ dv = dx, \quad v = x \end{array} \right) \\
 &= \cos(\ln x) \cdot x + \int \sin(\ln x) dx \\
 &\Rightarrow 2 \cdot \int \sin(\ln x) dx = \sin(\ln x) \cdot x - x \cdot \cos(\ln x) \\
 &\int \sin(\ln x) \cdot dx = \frac{x}{2} [\sin(\ln x) - \cos(\ln x)] + C \\
 &\quad \left( \begin{array}{l} \text{let } u = \ln x \\ \text{integral} = \int e^u \sin u du \end{array} \right) \quad \checkmark
 \end{aligned}$$

Figure 7: Solution to Section 8.2, problem 29

$$\begin{aligned}
 & \int x (\ln x)^2 dx \quad \text{let } u = \ln x \\
 & u = (\ln x)^2 \quad dv = x dx \quad \text{integral} = \int u^2 e^{2u} du \\
 & du = 2 \cdot \frac{1}{x} \ln x dx \quad v = \frac{x^2}{2} \\
 & \frac{x^2}{2} (\ln x)^2 - \int \frac{x^2}{2} \cdot \frac{2}{x} \ln x dx \\
 &= \frac{x^2}{2} (\ln x)^2 - \int x \ln x dx \\
 &= \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \ln x \cdot \frac{x^2}{2} - \int \frac{x}{2} dx = \ln x \cdot \frac{x^2}{2} - \frac{x^2}{4} \\
 & \frac{x^2}{2} (\ln x)^2 - \frac{x^2}{2} \ln x + \frac{x^2}{4} + C
 \end{aligned}$$

Figure 8: Solution to Section 8.2, problem 33

$$\begin{aligned}
 \int x \tan^{-1} x \, dx & \quad x dx = d\frac{x^2}{2} \\
 &= \frac{1}{2} x^2 \tan^{-1} x - \int \frac{1}{2} x^2 \cdot \left( \frac{1}{x^2+1} \right) dx \\
 &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{x^2+1} dx \\
 &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \left( 1 - \frac{1}{1+x^2} \right) dx \\
 &= \frac{1}{2} x^2 \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C
 \end{aligned}$$

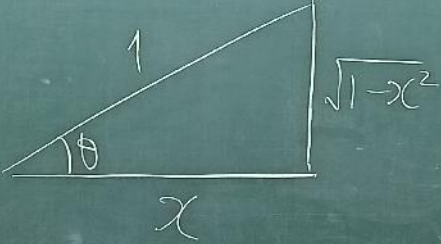
Figure 9: Solution to Section 8.2, problem 51

$$\begin{aligned}
 \text{8.2.71} \quad \int f^{-1}(x) dx &= x f^{-1}(x) - \int f(y) dy \\
 \int \sin^{-1} x \, dx + \int x \, d \sin^{-1} x &= x \sin^{-1} x \quad (y = f^{-1}(x)) \\
 \int x \, d \sin^{-1} x &= \int x \frac{1}{\sqrt{1-x^2}} dx = \int \frac{1}{2\sqrt{1-x^2}} dx^2 = \sqrt{1-x^2} + C \\
 u = \sin^{-1} x & \Rightarrow \int \sin u \, du = -\cos u = -\cos(\sin^{-1} x) + C \\
 \Rightarrow \int \sin^{-1} x \, dx &= x \sin^{-1} x + \sqrt{1-x^2} + C \\
 x \sin^{-1} x - \int x \, d \sin^{-1} x &+ \cos(\sin^{-1} x) \\
 \int \sin^{-1} x \, dx &= x \sin^{-1} x - \int \sin y \, dy = x \sin^{-1} x + \cos y + C \\
 &= x \sin^{-1} x + \cos(\sin^{-1} x) + C
 \end{aligned}$$

Figure 10: Solution to Section 8.2, problem 71



$$\int \cos^{-1} x \, dx = x \cos^{-1} x - \sin(\cos^{-1} x) + C$$

$$\stackrel{?}{=} x \cos^{-1} x - \sqrt{1-x^2} + C$$


A right triangle is drawn with a horizontal base of length  $x$ , a vertical height of length  $\sqrt{1-x^2}$ , and a hypotenuse of length 1. The angle between the base and the hypotenuse is labeled  $\theta$ .

$$\theta = \cos^{-1} x \Rightarrow \sin \theta = \sqrt{1-x^2}$$


$\therefore$  Yes, they are the same

Figure 11: Solution to Section 8.2, problem 75

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \ln \sec(\tan^{-1} x) + C$$

$$\stackrel{?}{=} x \tan^{-1} x - \ln \sqrt{1+x^2} + C$$

$\sec(\tan^{-1} x)$



A right triangle is drawn with a horizontal base of length 1, a vertical height of length  $x$ , and a hypotenuse of length  $\sqrt{x^2+1}$ . The angle between the base and the hypotenuse is labeled  $\theta$ .

$$\theta = \tan^{-1} x$$

$$\Rightarrow \sec(\tan^{-1} x) = \sec \theta$$

$$= \sqrt{1+x^2}$$

$$\therefore \ln \sec(\tan^{-1} x) = \ln \sqrt{1+x^2}$$

Figure 12: Solution to Section 8.2, problem 76