

Brief solutions to selected problems in homework 11

1. Section 6.2: Solutions, common mistakes and corrections:

$$\begin{aligned}
 V &= 2 \int_{z=0}^2 \pi r^2 dz = 2 \int_{z=0}^2 \pi \cdot \left(\sqrt{1 - \frac{z^2}{4}} \right)^2 dz \\
 &= 2\pi \int_{z=0}^2 \left(1 - \frac{z^2}{4} \right) dz \\
 &= 2\pi \cdot \left(z - \frac{1}{12} z^3 \right) \Big|_{z=0}^2 \\
 &= \frac{8}{3} \pi \text{ ft}^3 \quad \checkmark
 \end{aligned}$$

$x^2 + y^2 + \frac{z^2}{4} \leq 1$

Figure 1: Solution to homework 11, problem 2

2. Section 6.3: Solutions, common mistakes and corrections:

$$\begin{aligned}
 &\int_0^{\pi/6} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \\
 &\stackrel{\text{F.T.C.I}}{=} \int_0^{\pi/6} \sqrt{1 + \tan^2 t} dt \\
 &\stackrel{\substack{\text{sect} \geq 0 \\ \text{on } [0, \pi/6]}}{=} \int_0^{\pi/6} \sec t dt = \ln |\tan t + \sec t| \Big|_0^{\pi/6} \\
 &= \frac{1}{2} \ln 3 \quad \checkmark
 \end{aligned}$$

Figure 2: Solution to Section 6.3, problem 21

$$\begin{aligned}
 &\int_0^{\pi/6} \sqrt{1 + \tan^2 x} dx = \int_0^{\pi/6} \sec x dx \\
 &= \int_0^{\pi/6} \frac{1}{\cos x} dx = \int_0^{\pi/6} \frac{\cos x}{\cos^2 x} dx = \int_0^{\pi/6} \frac{\cos x}{1 - \sin^2 x} dx \\
 &\stackrel{u = \sin x}{=} \int_0^{\pi/6} \frac{du}{u(1-u^2)} = \int_0^{\pi/6} \frac{du}{u(1-u)(1+u)} \\
 &\stackrel{\substack{u+1 \rightarrow u+1 \\ (u-1)(u+1)}}{=} \int_0^{\pi/6} \frac{1}{u} du - \int_0^{\pi/6} \frac{1}{1-u} du + \int_0^{\pi/6} \frac{1}{1+u} du \\
 &= \left(\ln |u| - \ln |1-u| + \ln |1+u| \right) \Big|_0^{\pi/6} \\
 &= \frac{1}{2} \ln 3 \quad \checkmark
 \end{aligned}$$

Figure 3: Solution to Section 6.3, problem 21, another method

$$\frac{dy}{dx} = \frac{3}{2} (1 - x^{\frac{2}{3}})^{\frac{1}{2}} \left(-\frac{2}{3} x^{-\frac{1}{3}} \right) = -(1 - x^{\frac{2}{3}})^{\frac{1}{2}} x^{\frac{1}{3}}$$

$$\frac{dy^2}{dx^2} = x^{-\frac{2}{3}} (1 - x^{\frac{2}{3}})$$

$$L = \int_{\frac{\sqrt{2}}{4}}^1 \sqrt{1 + (x^{-\frac{2}{3}} (1 - x^{\frac{2}{3}}))} dx \cdot 8$$

$$= 8 \int_{\frac{\sqrt{2}}{4}}^1 x^{-\frac{1}{3}} dx$$

If $\frac{1}{3} = 0$, then $x^{\frac{1}{3}}$ not contin. on $[0, 1]$
 $\Rightarrow$ FTC2 can't be applied.

$$= 12 x^{\frac{2}{3}} \Big|_{\frac{\sqrt{2}}{4}}^1 = 6$$

Figure 4: Solution to Section 6.3, problem 26

29. $9x^2 = y(y-3)^2 \Rightarrow 18x dx = [(y-3)^2 + 2y(y-3)] dy$

$\Rightarrow 6x dx = (y-3)(y-1) dy$

$\Rightarrow 36x^2 dx^2 = (y-3)^2 (y-1)^2 dy^2$

$\Rightarrow dx^2 = \frac{(y-3)^2 (y-1)^2}{4 \cdot y(y-3)^2} dy^2$

$ds^2 = dx^2 + dy^2$

$= \frac{(y-1)^2}{4y} dy^2 + dy^2 = \left[\frac{(y-1)^2 + 4y}{4y} \right] dy^2 = \frac{(y+1)^2}{4y} dy^2$

Figure 5: Solution to Section 6.3, problem 29

3. Section 6.4: Solutions, common mistakes and corrections:

(a) $x = R \pm \sqrt{r^2 - y^2}$

(b) $x = R + r \cos t$, $y = r \sin t$

(c) $(R \pm \sqrt{r^2 - y^2}) = \pm \frac{1}{2} \frac{-2y}{\sqrt{r^2 - y^2}} = \mp \frac{y}{\sqrt{r^2 - y^2}}$

Diagram: A circle of radius r centered at $(R, 0)$ on the x-axis. The x-axis is labeled with $R - \sqrt{r^2 - y^2}$ and $R + \sqrt{r^2 - y^2}$.

$$\begin{aligned}
 &= \int_{-r}^r 2\pi(R + \sqrt{r^2 - y^2}) \sqrt{1 + \left(\frac{-y}{\sqrt{r^2 - y^2}}\right)^2} dy \\
 &= \int_{-r}^r \frac{2\pi r R}{\sqrt{r^2 - y^2}} dy + \int_{-r}^r 2\pi r \cancel{X} dy \Rightarrow \\
 &= 2 \int_{-r}^r \frac{2\pi r R}{\sqrt{r^2 - y^2}} dy = 4\pi r R \int_{-r}^r \frac{dy}{\sqrt{r^2 - y^2}} = \\
 &= 4\pi r R \left[\sin^{-1}\left(\frac{y}{r}\right) \right]_{-r}^r = 4\pi r R \left(\sin^{-1}\left(\frac{r}{r}\right) - \sin^{-1}\left(\frac{-r}{r}\right) \right) \\
 &= 4\pi^2 r R \quad \#
 \end{aligned}$$

Figure 6: Solution to homework 11, problem 5(a)

(b) $x = R + r \cos t$
 $y = r \sin t$

$$\begin{aligned}
 &\int_0^{2\pi} 2\pi x(t) \cdot \sqrt{(x'(t))^2 + (y'(t))^2} dt \\
 &\Rightarrow \int_0^{2\pi} 2\pi(R + r \cos t) r dt \\
 &= \int_0^{2\pi} 2\pi r R dt + \int_0^{2\pi} 2\pi r \cos t dt \\
 &= 2\pi r R(2\pi) + 2\pi r \sin t \Big|_0^{2\pi} \\
 &= 4\pi^2 r R \quad \#
 \end{aligned}$$

Figure 7: Solution to homework 11, problem 5(b)

$$\begin{aligned}
 23. \int 2\pi y \, ds &= \int 2\pi y \sqrt{dx^2 + dy^2} = \int 2\pi y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \, dy \\
 \Rightarrow S &= \int_{y=1}^{y=2} 2\pi y \sqrt{(x')^2 + 1} \, dy \\
 &= \int_{y=1}^{y=2} 2\pi y \sqrt{\left(y^3 - \frac{1}{4y^3}\right)^2 + 1} \, dy \quad (x' = y^3 - \frac{1}{4y^3}) \\
 &= \int_1^2 2\pi y \sqrt{\left(y^3 + \frac{1}{4y^3}\right)^2} \, dy \\
 &= \int_1^2 2\pi y \left[y^3 + \frac{1}{4y^3}\right] \, dy \quad (\because y^3 + \frac{1}{4y^3} \geq 0 \text{ if } y \in [1, 2]) \\
 &= 2\pi \left(\frac{y^5}{5} - \frac{1}{4y}\right) \Big|_1^2 = \frac{253}{20} \pi
 \end{aligned}$$

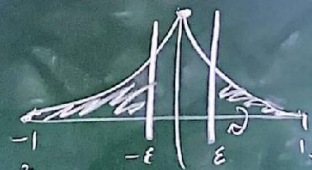
Figure 8: Solution to Section 6.4, problem 23

$$\begin{aligned}
 &\underline{6.4.32} \\
 S &= \int 2\pi y \, ds = \int 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \\
 \text{In 1st quadrant, } x &= (1 - y^{\frac{2}{3}})^{\frac{3}{2}} \quad (y > 0). \\
 \frac{dx}{dy} &\stackrel{(\text{check})}{=} -y^{\frac{1}{3}} (1 - y^{\frac{2}{3}})^{\frac{1}{2}} \Rightarrow \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \stackrel{(\text{check})}{=} \sqrt{1 + y^{\frac{2}{3}} - 1} = y^{\frac{1}{3}} \\
 \Rightarrow S &= \int_0^1 2\pi y (y^{\frac{1}{3}}) \, dy = \int_0^1 2\pi y^{\frac{7}{3}} \, dy \\
 &= \left[\frac{12\pi}{5} y^{\frac{5}{2}} \right]_0^1 = \frac{12\pi}{5} \#
 \end{aligned}$$

Figure 9: Solution to Section 6.4, problem 32, method 1

(Sol-2),

Note that



$$S = \lim_{\varepsilon \rightarrow 0^+} 2 \times \left(\text{Area of revolving } x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1 \text{ along } x\text{-axis} \right. \\ \left. \text{and } \varepsilon \leq x \leq 1 \right)$$

Since $2\pi(1-x^{\frac{2}{3}})^{\frac{3}{2}} x^{\frac{-1}{3}}$ conti. on $[\varepsilon, 1]$

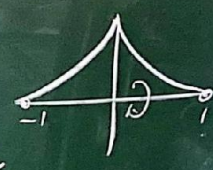
$$S = \lim_{\varepsilon \rightarrow 0^+} 2 \times \int_{\varepsilon}^1 2\pi(1-x^{\frac{2}{3}})^{\frac{3}{2}} x^{\frac{-1}{3}} dx$$

$$= 6\pi \lim_{\varepsilon \rightarrow 0^+} \left[\frac{2}{5} u^{\frac{5}{2}} \right]_{\varepsilon}^1 = 6\pi \cdot \left(\frac{2}{5} - \lim_{\varepsilon \rightarrow 0^+} \frac{2}{5} \varepsilon^{\frac{5}{2}} \right)$$

$$= \frac{12}{5} \pi - 0 = \frac{12}{5} \pi \quad \#$$

Figure 10: Solution to Section 6.4, problem 32, method 2

錯果寫法.

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1 \Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = x^{\frac{-1}{3}}$$


$$\Rightarrow S = 2 \times \int_0^1 2\pi(1-x^{\frac{2}{3}})^{\frac{3}{2}} x^{\frac{-1}{3}} dx$$

$$u = 1 - x^{\frac{2}{3}} \Rightarrow du = -\frac{2}{3} x^{\frac{-1}{3}} dx, \quad x^{\frac{-1}{3}} dx = -\frac{3}{2} du$$

$$\Rightarrow = -6\pi \int_1^0 u^{\frac{3}{2}} du = \frac{12}{5} \pi,$$

Problem: $2\pi(1-x^{\frac{2}{3}})^{\frac{3}{2}} x^{\frac{-1}{3}}$ not conti. on $[0, 1]$.

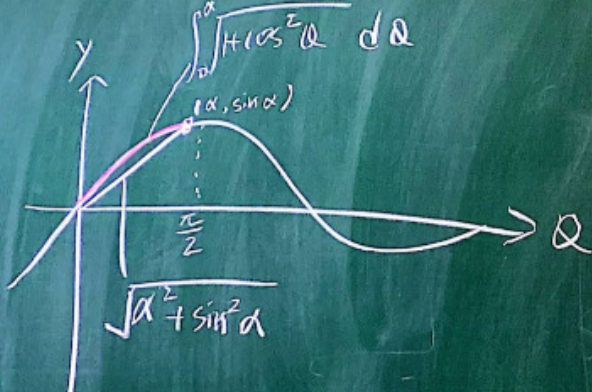
=> FTC2 can't be applied

Figure 11: Solution to Section 6.4, problem 32, method 2, continued

4. Chapter 6, additional and advanced problems: Solutions, common mistakes and corrections:

Ch6adv4

a. $\int_0^\alpha \sqrt{1 + \cos^2 Q} \, dQ = \int_0^\alpha \sqrt{1 + \left(\frac{d \sin Q}{dQ}\right)^2} \, dQ$



b. $\int_a^b \sqrt{1 + (f'(x))^2} \, dx \geq \sqrt{(b-a)^2 + (f(b) - f(a))^2}$

Figure 12: Solution to Chapter 6, additional and advanced problems: problem 4