

Brief solutions to selected problems in homework 07

1. Section 4.2: Solutions, common mistakes and corrections:

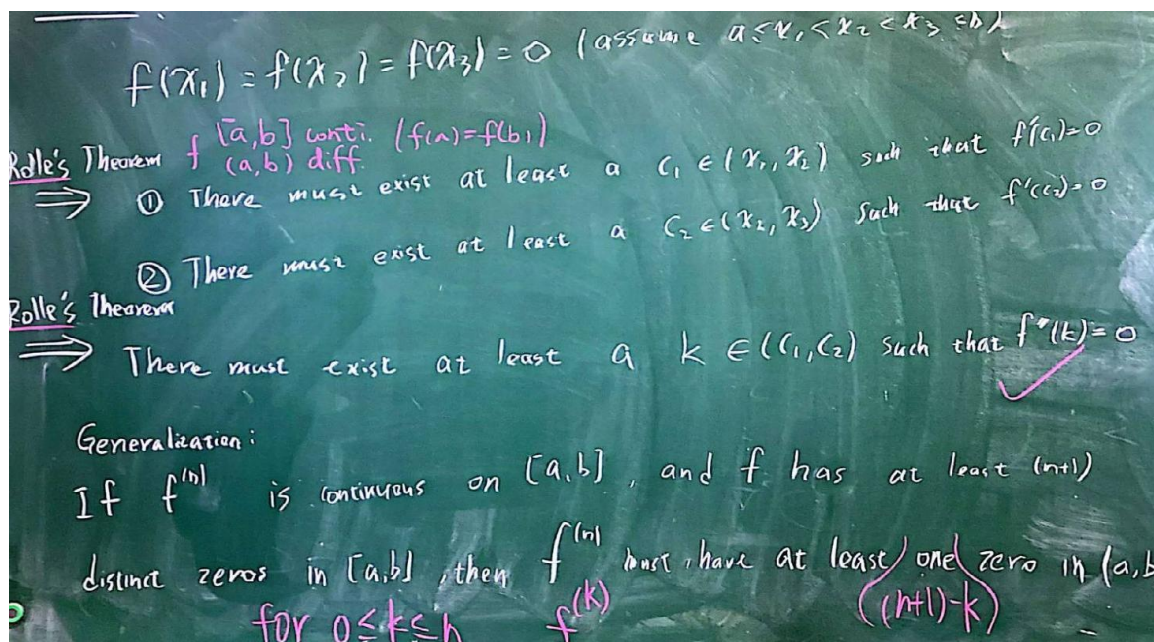


Figure 1: Solution to Section 4.2, problem 18

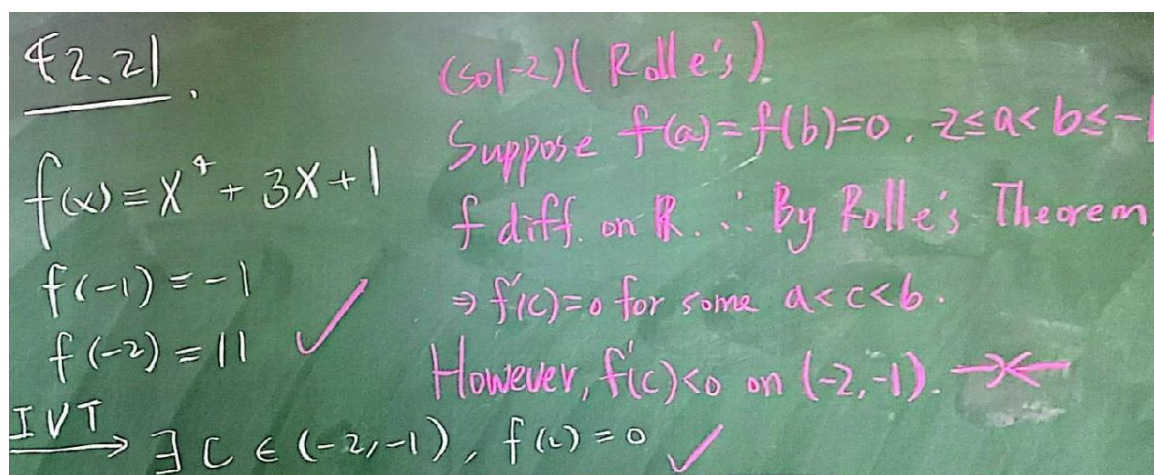


Figure 2: Solution to Section 4.2, problem 21

$f(a)f(b) < 0 \xrightarrow{\text{IVT}}$ at least one zero.
 If $f(x) = 0$ exists two roots
 $f(a_1) = f(b_1) \Rightarrow$ By Rolle's rule
 there exists
 $c \in (a_1, b_1)$
 $f'(c) = 0$. But in this case
 $f'(c) \neq 0$, contradiction.
 We have proved $f(x) = 0$ exactly
 once between a and b .

Figure 3: Solution to Section 4.2, problem 61

4.2.65, $|\cos x - 1| \leq |x|$
 Let $x \in \mathbb{R} - \{0\}$
 $f(t) = \cos t$
 $f'(t) = -\sin t$ By MVT, ($f(t)$ diff. on $\mathbb{R}$)
 $f(x) - f(0) = f'(c)(x - 0)$ for some c between x & 0 .
 $|\cos x - 1| = |-\sin c| |x|$
 $-1 \leq \sin c \leq 1$
 $|\sin c| \leq 1$
 $|\cos x - 1| \leq 1 \cdot |x|$
 (If $x = 0$, $|\cos 0 - 1| = 0$
 $|x| = 0$
 $\rightarrow$ inequality holds)

Figure 4: Solution to Section 4.2, problem 65

2. Section 4.3: Solutions, common mistakes and corrections:

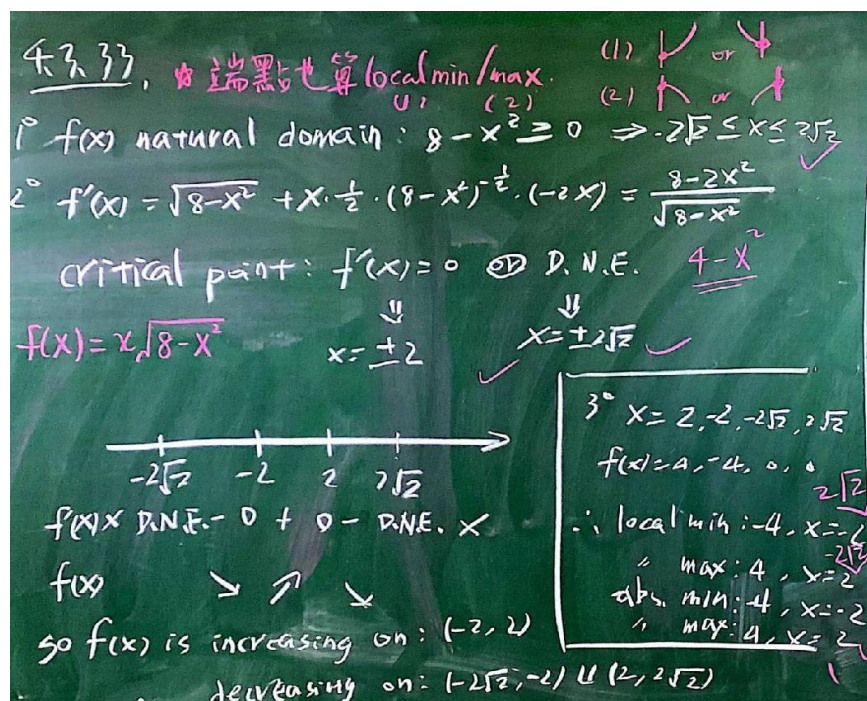


Figure 5: Solution to Section 4.3, problem 33

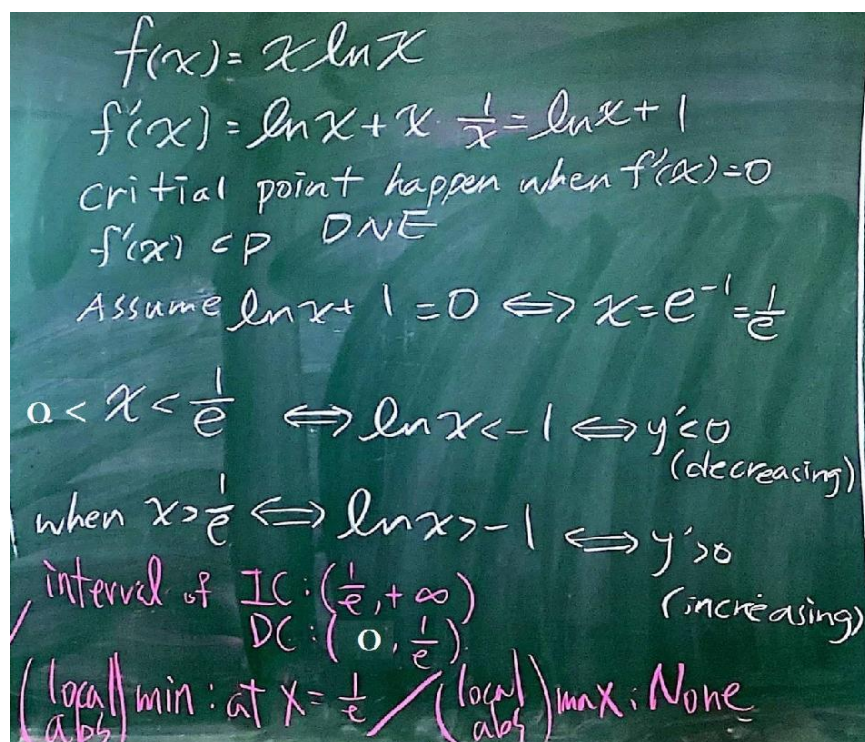


Figure 6: Solution to Section 4.3, problem 43

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$$f'(x) = 3ax^2 + 2bx + c$$

① $f(0) = 0$, $f(1) = -1$
 $d = 0$, $a + b + c = -1$

② $f'(1) = f'(0) = 0$
 $c = 0$, $3a + 2b = 0$

$\Rightarrow a = 2$, $b = -3$

Figure 7: Solution to Section 4.3, problem 74

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$$f'(x) = e^x - 2$$

$f'(x) = 0$ at $\ln(2)$

$f'(x) < 0$ on $[0, \ln(2))$

$f'(x) > 0$ on $(\ln(2), 1]$

f has an absolute minimum at $\ln(2)$

$f(0) = 1$, $f(1) = e - 2$

absolute maximum

Figure 8: Solution to Section 4.3, problem 77

3. Section 4.5: Solutions, common mistakes and corrections:

$$\lim_{x \rightarrow 0^+} \frac{(\ln x)^2}{\ln(\sin x)} \quad \left(\frac{\infty}{-\infty} \right)$$

or

$$\lim_{x \rightarrow 0^+} \frac{2 \ln x \cdot \frac{1}{x}}{\frac{\cos x}{\sin x}} \quad \left(\frac{-\infty}{\infty} \right) \xrightarrow{\text{L'H}} \lim_{x \rightarrow 0^+} \frac{2 \left(\frac{1 - \ln x}{x^2} \right)}{-\csc^2 x} \quad \left(\frac{\infty}{-\infty} \right)$$

= 繼續用 L'Hopital 只會更複雜 得不到結果

or

$$\lim_{x \rightarrow 0^+} \left(\frac{\sin x}{x} \right) \left(\frac{2 \ln x}{\cos x} \right) \quad \left(1 \cdot \frac{-\infty}{1} = -\infty \right)$$

or

$$\lim_{x \rightarrow 0^+} \frac{(2 \ln x) \sin x}{(x \cos x)} \quad \left(\frac{-\infty \cdot 0}{0} \right)$$

不知是否 $\frac{0}{0}$ 或 $\frac{\infty}{\infty}$ 不能直接用 L'Hopital

需先確認 $\lim_{x \rightarrow 0^+} (2 \ln x) \sin x = 0$, 故為 $\frac{0}{0}$

可再用 L'Hopital, $= \lim_{x \rightarrow 0^+} \frac{2 \left(\frac{\sin x}{x} + \cos x \ln x \right)}{\cos x - x \sin x} \quad \left(\frac{0}{1-0} = 0 \right)$

Figure 9: Solution to Section 4.5, problem 39

$$\lim_{x \rightarrow 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right) = \lim_{x \rightarrow 1^+} \frac{\ln x - x + 1}{(x-1) \ln x} \quad \frac{0}{0}$$

$$= \lim_{x \rightarrow 1^+} \frac{\frac{1}{x} - 1}{\ln x + \frac{x}{x} - \frac{1}{x}} \quad \frac{0}{0}$$

$$= \lim_{x \rightarrow 1^+} \frac{1-x}{x \ln x + x - 1} \quad \frac{0}{0} = \lim_{x \rightarrow 1^+} \frac{-1}{\ln x + 1 + 1} = -\frac{1}{2}$$

Figure 10: Solution to Section 4.5, problem 41

$$5. \lim_{x \rightarrow 0^+} \frac{e^{-\frac{1}{x}}}{x^k} = \lim_{x \rightarrow 0^+} \frac{e^{-\frac{1}{x}}}{e^{k \ln x}} = \lim_{x \rightarrow 0^+} e^{-\frac{1}{x} - k \ln x}$$

$$= e^{-\left(\lim_{x \rightarrow 0^+} \frac{1}{x} + k \ln x \right)}$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x} + k \ln x = \lim_{x \rightarrow 0^+} \frac{1 + k x \ln x}{x} = +\infty$$

$$\left(\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} \quad \left(\frac{-\infty}{\infty} \right) \xrightarrow{\text{L'H}} \lim_{x \rightarrow 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \rightarrow 0^+} -x = 0 \right)$$

$$\Rightarrow \text{ff } \# = e^{-\infty} = 0 \quad \# \quad \left(\text{This also proves } \lim_{x \rightarrow 0^+} \frac{e^{-\frac{1}{x}}}{x^k} = 0 \text{ for all } k \in \mathbb{R} \right)$$

Figure 11: Solution to Homework 07, Problem 5