

Brief solutions to selected problems in homework 06

1. Section 3.11: Solutions, common mistakes and corrections:

53

$$f(x) = x^2 \cdot \pi \cdot 30$$

$$f'(x) = 2x \cdot \pi \cdot 30$$

$$dy = f'(x) \cdot dx$$

$x = 5.5$

$dx = 0.5$

~~$= 660 \pi \cdot 5.5 \cdot dx$~~

$= 330 \pi \cdot 5.5 \cdot 0.5$

Figure 1: Solution to Section 3.11, problem 53 and some corrections

Remark:

" $dy = f'(a)dx$ " is in the notation of "differential" which we skipped.

In the notation of linear approximation, the idea is:

Let $L(x) = f(a) + f'(a)(x - a)$ be the linear approximation of $f(x)$ near $x = a$,

$$f(x) \approx L(x) \implies \Delta f \approx \Delta L$$

where $\Delta f = f(x) - f(a)$ and $\Delta L = L(x) - L(a)$. It is easy to check by direct calculating that $L(x) - L(a) = f'(a)\Delta x$ where $\Delta x = x - a$.

Therefore

$$\Delta f \approx \Delta L = f'(a)\Delta x$$

Here $f(x) = \pi x^2 h = 30\pi x^2$, $a = 5.5$ and $\Delta x = 0.5$.

Note: $dy = f'(a)dx$ is just another way of saying $\Delta L = f'(a)\Delta x$.

3.11.55.
(sol-1: linearization)

$$f(x) \approx f(r) + f'(r)(x-r) \Rightarrow \underbrace{f(x) - f(r)}_{\Delta f} \approx f'(r) \underbrace{(x-r)}_{\Delta x}$$

實際 (actual) 預測 (predicted)
 $\Delta x = x - r$

$$\Rightarrow \frac{\Delta f}{f(r)} \approx \frac{f'(r) \Delta x}{f(r)} \quad \text{Assumption} \Rightarrow \left| \frac{\Delta x}{r} \right| \leq 2\%$$

目標: find the range of $\frac{|\Delta f|}{f(r)}$

(a) $f(x) = 2\pi x \Rightarrow f'(x) = 2\pi \Rightarrow \frac{\Delta f}{f(r)} \approx \frac{2\pi}{2\pi r} \Delta x \Rightarrow \frac{|\Delta f|}{f(r)} \approx \frac{|\Delta x|}{r} \leq (2\%)$

(b) $g(x) = \pi x^2 \Rightarrow g'(x) = 2\pi x \Rightarrow \frac{\Delta g}{g(r)} = \frac{2\pi r}{\pi r^2} \Delta x \Rightarrow \frac{\Delta g}{g(r)} \approx \frac{2|\Delta x|}{r} \leq (4\%)$

Figure 2: Solution to Section 3.11, problem 55. Note: Δg near the end should be $|\Delta g|$ instead

3.11 66

1. $E(a) = 0$
 $E(a) = f(a) - m(a-a) - C = 0 \quad \begin{cases} 3^\circ \\ g(x) = m(x-a) + C \\ = f'(a)(x-a) + f(a) \end{cases}$
 $\rightarrow f(a) = C$

2° $\lim_{x \rightarrow a} \frac{f(x) - m(x-a) - C}{x-a} = 0$
 $\lim_{x \rightarrow a} \left(\frac{f(x) - C}{x-a} - m \right) = 0$
 $\rightarrow f(a) - m = 0$
 $m = f'(a)$

✓

Figure 3: Solution to Section 3.11, problem 66

2.

1° Let $f(x) = (1+x)^{\frac{1}{3}}$. $\therefore L(x) = 1 + \frac{1}{3}x$ at $x = \frac{0}{a}$ linearization

$x = 0.009$ decreasing

$f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$

$f''(x) = -\frac{2}{9}(1+x)^{-\frac{5}{3}} \rightarrow |f''| = \frac{2}{9}(1+x)^{-\frac{5}{3}}$

2° Based on (4), $|f(x) - L(x)| \leq \frac{1}{2} |f''(c)| \cdot (x-0)^2$

$c \in [0, 0.009]$ ($|f''(c)|$ has a maximum at $c=0$)

$\leq \frac{1}{2} \times \frac{2}{9} \times (0.009)^2$

$= 9 \times 10^{-6}$ ✓

Figure 4: Solution to Homework 06, problem 2

2. Section 4.1: Solutions, common mistakes and corrections:

$y = x^2 + \frac{2}{x}$

$y' = 2x + \left(\frac{-2}{x^2}\right)$

when $y' = 0 \Rightarrow 2x + \left(\frac{-2}{x^2}\right) = 0$

$\Rightarrow x = 1$

critical point = $(1, 3)$ ✓

y' DNE $\Rightarrow x = 0$. But $y(0)$ DNE, so $x = 0$ is not critical point.

Figure 5: Solution to Section 4.1, problem 49

4.1.67. $y = \cos^{-1}(x^2)$

~~$-1 \leq \cos^{-1}(x^2) \leq 1$~~

$-1 \leq x^2 \leq 1$

$-1 \leq x^2 \quad \forall x \in \mathbb{R}, \quad x^2 \leq 1, \quad |x| \leq 1$

$x \in [-1, 1] = \text{dom}(y)$

$y' = -\frac{2x}{\sqrt{1-x^4}}, \quad y' = 0 \Rightarrow 2x = 0, \quad x = 0$

y' is not defined $\Rightarrow \sqrt{1-x^4} = 0 \Rightarrow x^4 = 1$

$x = 0, \quad y = \frac{\pi}{2}$ abs. max

$x = -1, 1, \quad y = 0$ abs. min ✓ #

Figure 6: Solution to Section 4.1, problem 67

Figure 7: Solution to Section 4.1, problem 67. Note: $x = \pm 1$ are under consideration not because $f'(\pm 1)$ do not exist, but because $x = \pm 1$ are boundary points of domain of f .