

Brief solutions to selected problems in homework 05

1. Problem 1:

$$\begin{aligned}
 & f(f^{-1}(x)) = x \\
 dx \left[\begin{aligned} & f'(f^{-1}(x))(f^{-1})'(x) = 1 \\ & f''(f^{-1}(x))(f^{-1})'(x)(f^{-1})'(x) + f'(f^{-1}(x))(f^{-1})''(x) = 0 \end{aligned} \right. \\
 & (f^{-1})''(x) = \frac{-f''(f^{-1}(x))[(f^{-1})'(x)]^2}{f'(f^{-1}(x))} \\
 & \quad = \frac{-f''(f^{-1}(x))}{[f'(f^{-1}(x))]^3}
 \end{aligned}$$

Figure 1: Solution to Homework 05, problem 1, method 1

$$\begin{aligned}
 & \text{Let } f^{-1}(y) \text{ be the inverse function of } f(x) \\
 & \frac{d}{dy} f^{-1}(y) = \frac{1}{\frac{d}{dx} f(x) \Big|_{x=f^{-1}(y)}} = \frac{1}{f'(f^{-1}(y))} \\
 & \frac{d^2}{dy^2} f^{-1}(y) = \frac{-f''(f^{-1}(y)) \cdot \frac{1}{f'(f^{-1}(y))}}{[f'(f^{-1}(y))]^2} = \frac{-f''(f^{-1}(y))}{[f'(f^{-1}(y))]^3}
 \end{aligned}$$

Figure 2: Solution to Homework 05, problem 1, method 2

2. Section 3.8: Solutions, common mistakes and corrections:

3.8.39 $5 \ln(x^2+1) - \frac{1}{2} \ln(1-x)$

$$y = \ln \left(\frac{(x^2+1)^5}{\sqrt{1-x}} \right)$$

$$\frac{\sqrt{1-x} \cdot 5(x^2+1)^4 \cdot 2x + \frac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1)}{1-x}$$

$$= \frac{10x(x^2+1)^4(1-x)^{\frac{1}{2}} + \frac{1}{2}(1-x)^{-\frac{1}{2}}}{x-1}$$

$$y' = 5 \frac{2x}{x^2+1} - \frac{1}{2} \frac{-1}{1-x} \quad \#$$

Figure 3: Solution to Section 3.8, problem 39

3.8.65, $x^y = y^x$

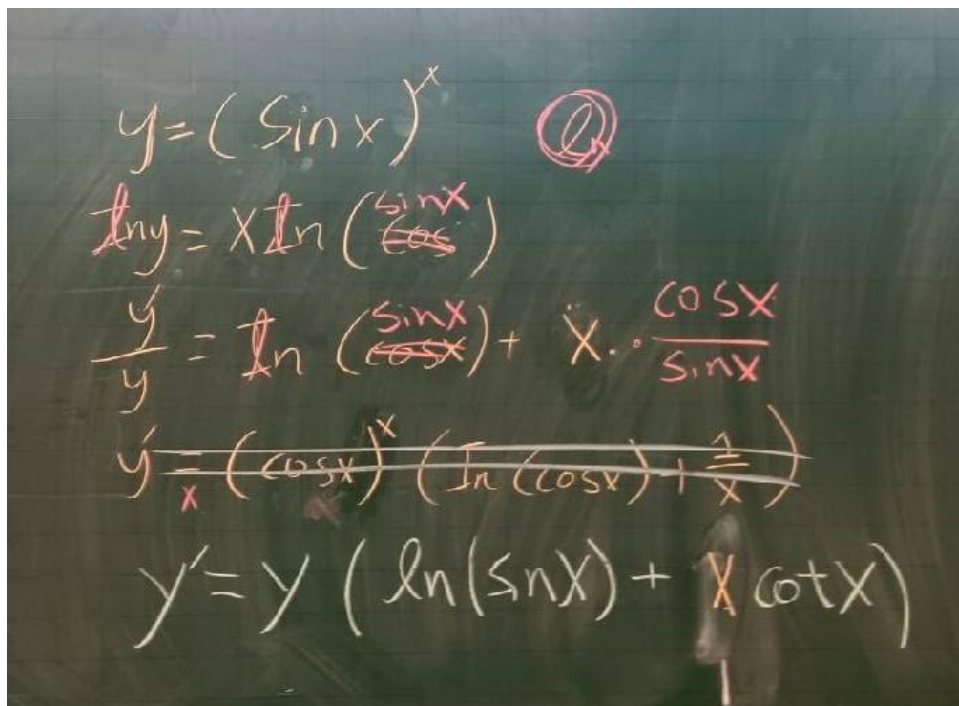
$$\ln x^y = \ln y^x \quad y \cdot \ln x = x \cdot \ln y$$

$$y' \cdot \ln x + y \cdot (\ln x)' = x' \cdot \ln y + x \cdot (\ln y)'$$

$$y' \cdot \ln x + y \cdot \frac{1}{x} = \ln y + x \cdot \frac{1}{y} \cdot y'$$

$$y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} = \frac{xy \cdot \ln y - y^2}{xy \cdot \ln x - x^2} = \frac{y}{x} \left(\frac{x \ln y - y}{y \ln x - x} \right)$$

Figure 4: Solution to Section 3.8, problem 65



$$y = (\sin x)^x \quad \textcircled{Q}$$

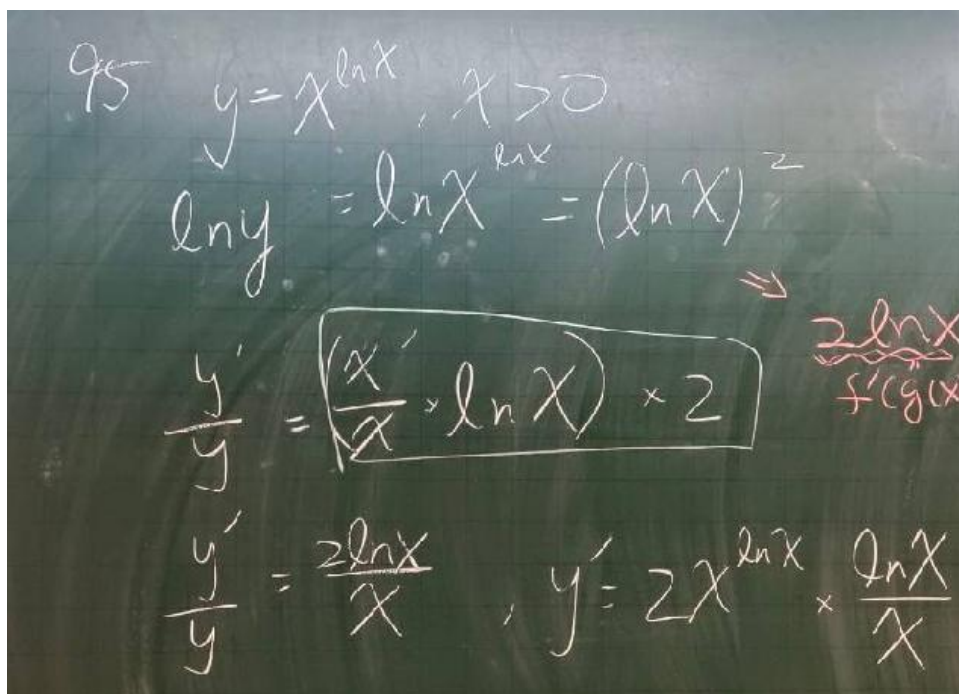
$$\ln y = x \ln(\sin x)$$

$$\frac{y}{y'} = \ln(\sin x) + x \cdot \frac{\cos x}{\sin x}$$

$$\cancel{y'} = \cancel{(\cos x)^x} \left(\ln(\cos x) + \frac{1}{x} \right)$$

$$y' = y (\ln(\sin x) + x \cot x)$$

Figure 5: Solution to Section 3.8, problem 93. Alternative method: start with $y = (e^{\ln \sin x})^x = e^{x \ln \sin x}$



$$95 \quad y = x^{\ln x}, \quad x > 0$$

$$\ln y = \ln x^{\ln x} = (\ln x)^2$$

$$\frac{y}{y'} = \left(\frac{x'}{x} \cdot \ln x \right) \times 2 \quad \Rightarrow \quad \frac{2 \ln x}{f'(g(x))}$$

$$\frac{y}{y'} = \frac{2 \ln x}{x}, \quad y' = 2x^{\ln x} \times \frac{\ln x}{x}$$

Figure 6: Solution to Section 3.8, problem 95. Alternative method: start with $y = (e^{\ln x})^x = e^{(\ln x)^2}$

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Show that $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for $x > 0$

Pf.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^{\left(x \lim_{n \rightarrow \infty} \frac{n \ln\left(1 + \frac{x}{n}\right)}{x}\right)} = e^x \lim_{t \rightarrow 0^+} \frac{\ln(1+t) - \ln(1)}{t}$$

$$= \lim_{n \rightarrow \infty} e^{n \ln\left(1 + \frac{x}{n}\right)} = e^x \lim_{n \rightarrow \infty} \frac{\ln\left(1 + \frac{x}{n}\right)}{\frac{x}{n}}$$

Since e^x is constant

$$= e^x \lim_{t \rightarrow 0^+} \frac{\ln(1+t)}{t}$$

$\left. \frac{\ln(1+t)}{t} \right|_{t=1} = 1$
 $\left. \frac{1}{t} \right|_{t=1} = 1$
 $\left. \frac{1}{t} \right|_{t=1} = 1$

Figure 7: Solution to Section 3.8, problem 98

3. Section 3.9: Solutions, common mistakes and corrections:

3.9.53. Suppose $\sec^{-1} x = y$, $\sec y = x$

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{\sec'(\sec^{-1} x)}$$

$$= \frac{1}{\sec(\sec^{-1} x) \tan(\sec^{-1} x)}$$

$\neq \frac{1}{|x| \sqrt{\tan y}} = \frac{1}{|x| \sqrt{x^2 - 1}}$

$\sec(\sec^{-1} x) = x$
 $\tan(\sec^{-1} x) = \begin{cases} \sqrt{x^2 - 1}, & x \geq 1 \\ -\sqrt{x^2 - 1}, & x < -1 \end{cases} \cdot \tan y + 1 = \sec^2 y$

$$\Rightarrow \sec(\sec^{-1} x) \cdot \tan(\sec^{-1} x) = |x| \sqrt{x^2 - 1}$$

Figure 8: Solution to Section 3.9, problem 53

$$\begin{aligned}
 g(x) &= 2 \tan^{-1} \sqrt{x} \\
 g'(x) &= 2 \cdot \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{1}{2} x^{-\frac{1}{2}} \\
 &= \frac{1}{(1+x)\sqrt{x}} \\
 f(x) &= \sin^{-1} \frac{x-1}{x+1}, \quad x \geq 0 \\
 f'(x) &= \frac{1}{\sqrt{1 - \left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{(x+1) - (x-1)}{(x+1)^2} \\
 &= \frac{2}{\sqrt{(x+1)^4 - (x-1)^2 (x+1)^2}} = \frac{2}{2(x+1)\sqrt{x}} = \frac{1}{(x+1)\sqrt{x}}
 \end{aligned}$$

$\Rightarrow f(x) = g(x) + c$
 for some constant c .
 $x=0 \Rightarrow \lambda \Rightarrow c = \frac{\pi}{2}$
 $\Rightarrow f(x) = g(x) - \frac{\pi}{2} \Rightarrow f'(x) = g'(x)$

Figure 9: Solution to Section 3.9, problem 55

4. Chapter 3, additional and advanced problems: Solutions, common mistakes and corrections:

$$\begin{aligned}
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 \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} &= 0 = f(0) \\
 f'(0) &= \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\frac{1 - \cos x}{x} - 0}{x} \\
 &= \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x^2} \right) \left(\frac{1 + \cos x}{1 + \cos x} \right) \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \left(\frac{1}{1 + \cos x} \right) = \frac{1}{2} \\
 \therefore f(x) &\text{ is } \cancel{\text{continuous}} \text{ at } 0 \quad \checkmark \\
 &\text{diff}
 \end{aligned}$$

Figure 10: Solution to Chapter 3, additional and advanced problems: problem 16

$$\begin{aligned}
 & 2/ \quad f(x)g(x) - f(x_0)g(x_0) \\
 & = f(x)g(x) - f(x_0)g(x) + \underbrace{f(x_0)g(x) - f(x_0)g(x_0)}_{=0 \text{ since } f(x_0)=0} \\
 & \lim_{x \rightarrow x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\
 & = \lim_{x \rightarrow x_0} \frac{g(x)[f(x) - f(x_0)]}{x - x_0} + \lim_{x \rightarrow x_0} \underbrace{\frac{f(x_0)[g(x) - g(x_0)]}{x - x_0}}_{\text{crossed out}} \\
 & = \lim_{x \rightarrow x_0} g(x) \cdot f'(x) \quad \text{crossed out} = g(x_0) f'(x) \quad \because g(x) \text{ is diff}
 \end{aligned}$$

Figure 11: Solution to Chapter 3, additional and advanced problems: problem 21

Chap adv 23

By 22(d), $h(x) = \begin{cases} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & x = 0 \end{cases} \rightarrow 22(d)$

$\lim_{x \rightarrow 0} 2x \sin(\frac{1}{x}) = 0$ but $\lim_{x \rightarrow 0} \cos(\frac{1}{x}) \text{ DNE} \Rightarrow \lim_{x \rightarrow 0} h(x) \neq h'(0)$
DNE 0

$\Rightarrow h(x)$ is discontinuous at $x=0$ #

For $k = xh(x)$, $k'(x) = h(x) + xh'(x) \Rightarrow k'(0) = h(0) = 0$ exists.

Moreover, $\lim_{x \rightarrow 0} k'(x) = \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} + \lim_{x \rightarrow 0} x \left[2x \sin \frac{1}{x} - \cos \frac{1}{x} \right]$

By Squeeze Theorem, $\lim_{x \rightarrow 0} x \cos(\frac{1}{x}) = 0$

$\Rightarrow \lim_{x \rightarrow 0} k'(x) = \lim_{x \rightarrow 0} 3x^2 \sin(\frac{1}{x}) - \lim_{x \rightarrow 0} x \cos(\frac{1}{x}) = 0 = k'(0)$

$\Rightarrow k'(x)$ is continuous at $x=0$ #

Figure 12: Solution to Chapter 3, additional and advanced problems: problem 23