Brief solutions to selected problems in homework 04

1. Section 3.5: Solutions, common mistakes and corrections:

   Problem 58:

   Continuity: We have $\lim_{x\to 0^+} g(x) = g(0) = 1$ and $\lim_{x\to 0^-} g(x) = b$. Therefore $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x) = g(0)$ implies g is continuous at $x = 0$ if and only if $b = 1$.

   Differentiability: $\lim_{x\to 0^+} \dfrac{g(x) - g(0)}{x - 0} = 0$, $\lim_{x\to 0^-} \dfrac{g(x) - g(0)}{x - 0} = 1$ for any $b \in \mathbb{R}$, therefore $\lim_{x\to 0} \dfrac{g(x) - g(0)}{x - 0}$ does not exist. g is not differentiable at $x = 0$ for any $b \in \mathbb{R}$.

2. Problem 2:

   (a): Yes, since $\lim_{x\to 0} f(x) = f(0) = 1$.

   (b): We show that $\lim_{x\to 0} \dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right) = 0$.

   $\dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right) = \dfrac{x\cos x - \sin x}{x^2} = \dfrac{\cos x - \dfrac{\sin x}{x}}{x}$.

   From the inequality (page 104 of the textbook)

   $$1 > \frac{\sin x}{x} > \cos x, \quad \text{on } 0 < x < \frac{\pi}{2},$$

   which also holds for $0 > x > \dfrac{-\pi}{2}$ (since both $\dfrac{\sin x}{x}$ and $\cos x$ are even functions), we see that if $0 < |x| < \dfrac{\pi}{2}$, then

   $$\left|\frac{d}{dx}\left(\frac{\sin x}{x}\right)\right| = \left|\frac{\cos x - \dfrac{\sin x}{x}}{x}\right| = \frac{\dfrac{\sin x}{x} - \cos x}{|x|} < \frac{1 - \cos x}{|x|} = \frac{2\sin^2\dfrac{x}{2}}{|x|} < \frac{2\left(\dfrac{x}{2}\right)^2}{|x|} = \frac{|x|}{2}$$

   Overall, we have

   $$0 \le \left|\frac{d}{dx}\left(\frac{\sin x}{x}\right)\right| < \frac{|x|}{2}, \quad \text{on } 0 < |x| < \frac{\pi}{2}.$$

   Since $\lim_{x\to 0} \dfrac{|x|}{2} = 0$, it follows from the Sandwich Theorem that $\lim_{x\to 0} \left|\dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right)\right| = 0$, and therefore $\lim_{x\to 0} \dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right) = 0$.

   The proof for $f'(0) = \lim_{x\to 0} \dfrac{\dfrac{\sin x}{x} - 1}{x - 0} = 0$ is similar.

3. Problem 4:

   Ans: $= f'(g(2)) \cdot g'(2) = f'(3) \cdot g'(2) = 0.4$

4. Section 3.7: Solutions, common mistakes and corrections:

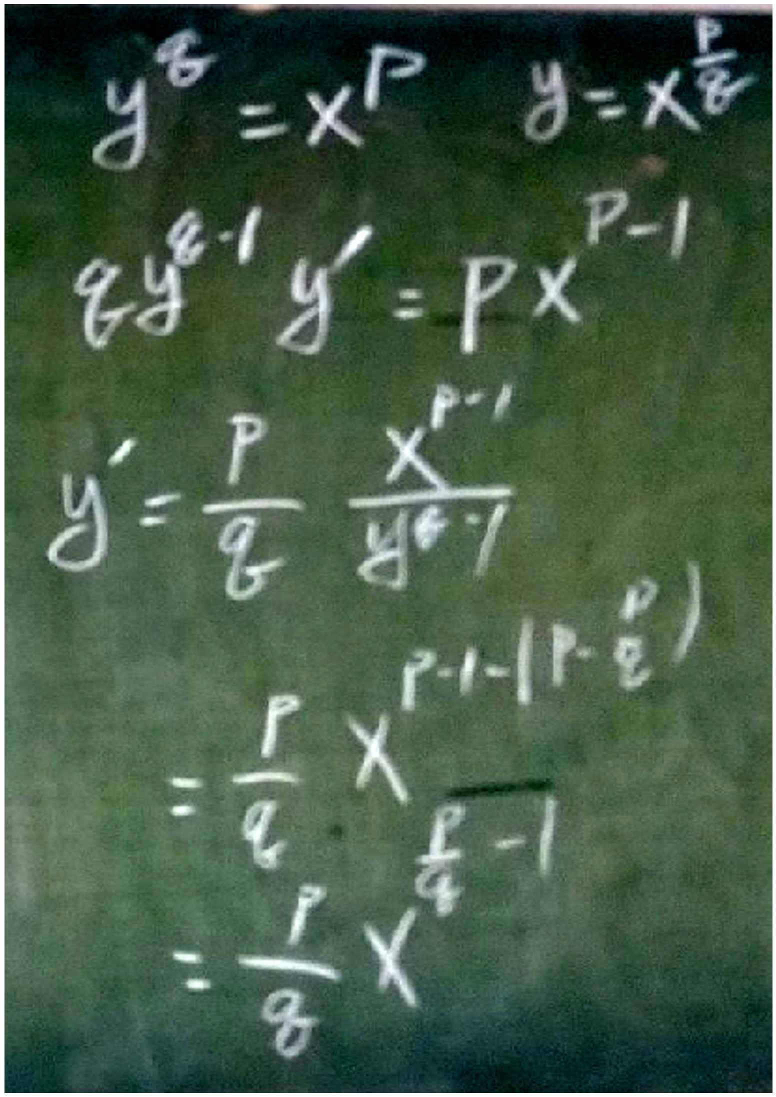

Figure 1: Solution to Section 3.7, problem 48

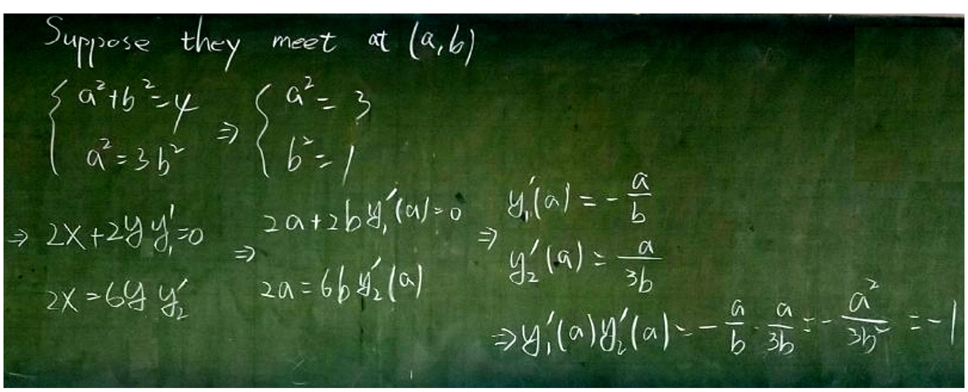

Figure 2: Solution to Section 3.7, problem 51(a)