

## Brief solutions to selected problems in homework 03

### 1. Section 2.6: Solutions, common mistakes and corrections:

Sec 2.6

61. For any  $M > 0$ , take  $\delta = \left(\frac{1}{M}\right)^{\frac{3}{2}} > 0$  s.t.

(a) if  $0 < x < \delta$  then  $f(x) = \frac{1}{x^{\frac{3}{2}}} > \frac{1}{\delta^{\frac{3}{2}}} = M$

(b) if  $0 < -x < \delta$  then  $f(x) = \frac{1}{x^{\frac{3}{2}}} > \frac{1}{\delta^{\frac{3}{2}}} = M$

(c) if  $0 < x-1 < \delta$  then  $f(x) = \frac{2}{(x-1)^{\frac{3}{2}}} > \frac{2}{\delta^{\frac{3}{2}}} = 2M > M$

(d) if  $0 < 1-x < \delta$  then  $f(x) = \frac{2}{(x-1)^{\frac{3}{2}}} > \frac{2}{\delta^{\frac{3}{2}}} = 2M > M$

85.  $\lim_{x \rightarrow \infty} (\sqrt{x^2+3x} - \sqrt{x^2-2x}) = \lim_{x \rightarrow \infty} \frac{5x}{\sqrt{x^2+3x} + \sqrt{x^2-2x}}$

$= \lim_{x \rightarrow \infty} \frac{5}{\sqrt{1+\frac{3}{x}} + \sqrt{1-\frac{2}{x}}}$

$\left( \lim_{x \rightarrow \infty} \frac{1}{x} = 0 \right) = \frac{5}{2}$

92. For any  $M > 0$ , take  $\delta = \sqrt{\frac{1}{M}} > 0$ ,

$0 < |x+5| < \delta \Rightarrow 0 < |x+5|^2 < \delta^2 = \frac{1}{M} \Rightarrow \frac{1}{(x+5)^2} > M$

93. (a)  $\lim_{x \rightarrow c} f(x) = \infty \Leftrightarrow$  For any  $M > 0$ , there exists a  $\delta > 0$  s.t.

if  $-\delta < x-c < 0$  then  $f(x) > M$

(b)  $\lim_{x \rightarrow c} f(x) = -\infty \Leftrightarrow$  For any  $-M < 0$ , there exists a  $\delta > 0$  s.t.

if  $0 < x-c < \delta$  then  $f(x) < -M$

(c)  $\lim_{x \rightarrow c} f(x) = -\infty \Leftrightarrow$  For any  $-M < 0$ , there exists a  $\delta > 0$  s.t.

if  $-\delta < x-c < 0$  then  $f(x) < -M$

Figure 1: Solution to selected problems in Section 2.6, part 1

95. For any  $-M < 0$ , take  $\delta = \frac{1}{M} > 0$  s.t.  
 if  $-\delta < x < 0$  then  $\frac{1}{x} < \frac{1}{-\delta} = -M$

97. For any  $M > 0$ , take  $\delta = \frac{1}{M} > 0$  s.t.  
 if  $0 < x-2 < \delta$  then  $\frac{1}{x-2} > \frac{1}{\delta} = M$

Figure 2: Solution to selected problems in Section 2.6, part 2

$\lim_{x \rightarrow \infty} f(x) = \infty \Leftrightarrow$  for any  $M > 0$ , there exists a  $N > 0$  s.t.  
 if  $N < x$  then  $f(x) > M$

$\lim_{x \rightarrow \infty} f(x) = -\infty \Leftrightarrow$  for any  $M > 0$ , there exists a  $N > 0$  s.t.  
 if  $N < x$  then  $f(x) < -M$

$\lim_{x \rightarrow -\infty} f(x) = \infty \Leftrightarrow$  for any  $M > 0$ , there exists a  $N > 0$  s.t.  
 if  $x < -N$  then  $f(x) > M$

$\lim_{x \rightarrow -\infty} f(x) = -\infty \Leftrightarrow$  for any  $M > 0$ , there exists a  $N > 0$  s.t.  
 if  $x < -N$  then  $f(x) < -M$

Show:  $\lim_{x \rightarrow \infty} -x^3 = -\infty$

for any  $M > 0$ , there exists a  $N = \sqrt[3]{M} > 0$  s.t.  
 if  $x > N$  then  $x^3 > N^3 = M$   
 $\Rightarrow -x^3 < -M$

Figure 3: Solution to Homework 03, problem 2

2. Section 3.2: Solutions, common mistakes and corrections:

Note:  
 $f$  is diff at  $x=0$   
 If  $\lim_{x \rightarrow 0} \frac{f(x)-f(0)}{x-0}$  exists

Step 1:  
 Find  $f(0)$   
 Since our assumption  $|f(x)| \leq x^2$  for  $-1 \leq x \leq 1$ ,  
 $|f(0)| \leq 0^2 = 0 \Rightarrow \boxed{f(0) = 0}$

Step 2:  
 Consider  $\lim_{x \rightarrow 0} \left| \frac{f(x)-f(0)}{x-0} \right|$   
 Fact: If  $\lim_{x \rightarrow 0} |g(x)| = 0$ , then  $\lim_{x \rightarrow 0} g(x) = 0$   
 $\Rightarrow \lim_{x \rightarrow 0} \left| \frac{f(x)-f(0)}{x-0} \right| = \lim_{x \rightarrow 0} \left| \frac{f(x)}{x} \right| \leq \lim_{x \rightarrow 0} \frac{x^2}{|x|} = \lim_{x \rightarrow 0} |x| = 0$   
 $\Rightarrow \lim_{x \rightarrow 0} \frac{f(x)-f(0)}{x-0} = 0$

Hence, by Fact, we have  
 $\lim_{x \rightarrow 0} \frac{f(x)-f(0)}{x-0} = 0$   
 $\Rightarrow f$  is diff at  $x=0$   
 $\Rightarrow f'(0) = 0$

Figure 4: Solution to Section 3.2, problem 58(a). Step 1: show that  $f(0) = 0$ . Step 2: show that  $f'(0) = 0$  as in problem 58(b)

3.2(58)

$|f(x)| \leq x^2$  for  $-1 \leq x \leq 1$

since  $x^2|_{x=-1} = 1$ ,  
 $x^2|_{x=1} = 1$   
 $x^2|_{x=0} = 0$ ,  
 and  $x^2$  is continuous on  $\mathbb{R}$   
 then  $f(x)$  is continuous & thus  
differentiable at  $x=0$  ;  $\times$   
 since  $x^2$ ,  $f(x)$  is mirrored  
 at  $x=0$ ,  
 then  $f'(0) = 0$

Figure 5: Common mistakes to Section 3.2, problem 58(a)



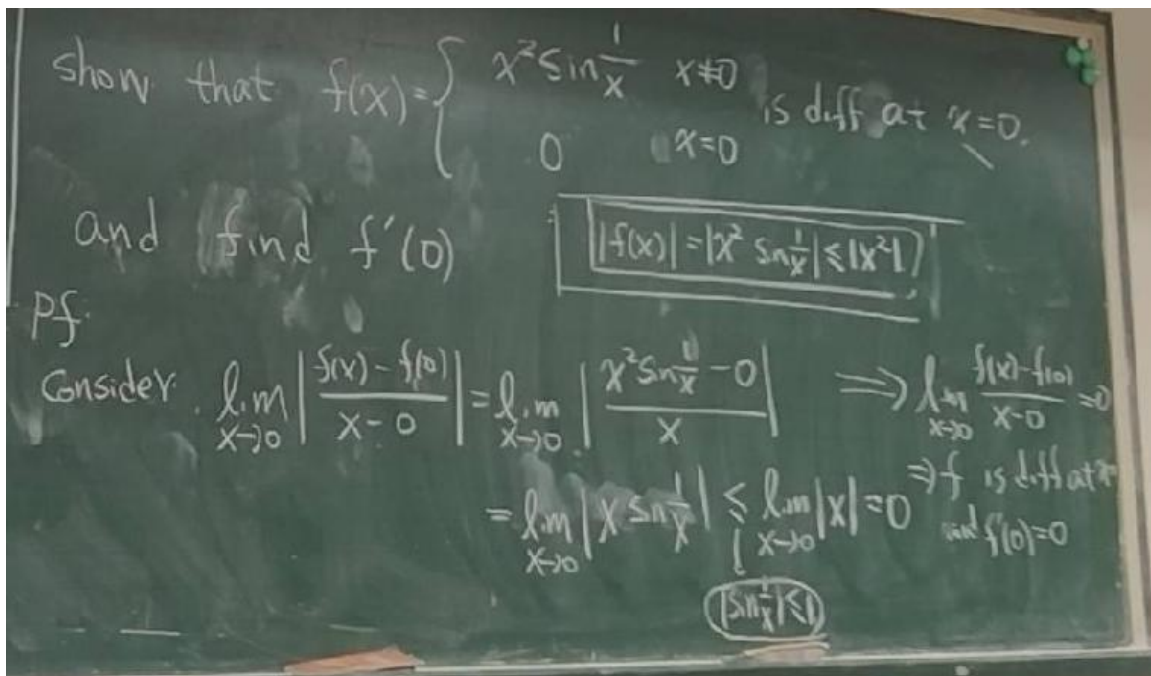


Figure 6: Solution to Section 3.2, problem 58(b)

3. Section 3.3: Solutions, common mistakes and corrections:

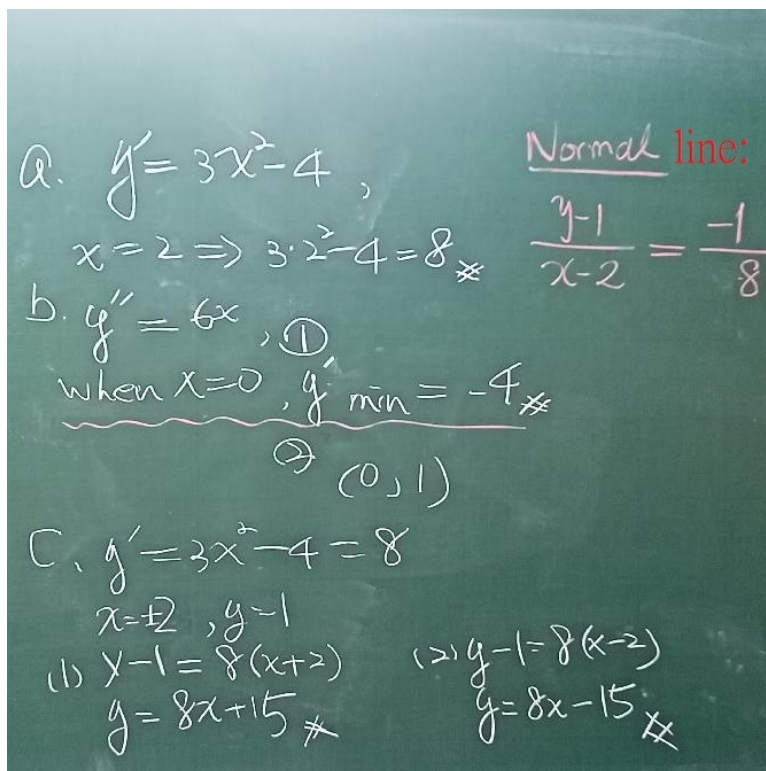


Figure 7: Solution to Section 3.3, problem 55

3.3 (67)

$$\lim_{x \rightarrow 1} \frac{(x-1)(x^{49} + x^{48} + \dots + 1)}{x-1}$$

$$= \lim_{x \rightarrow 1} x^{49} + x^{48} + \dots + 1 = 50 \quad \checkmark$$

$$\lim_{x \rightarrow 1} \frac{x^{50} - 1}{x - 1}$$

$$= \left. \frac{d}{dx} x^{50} \right|_{x=1}$$

$$= 50x^{49} \Big|_{x=1} = 50$$

Figure 8: A trick for Section 3.3, problem 67

3.3(70)

$f(x)$  is differentiable for all  $x$ -values, that is,  $f(x)$  is continuous on  $\mathbb{R}$ .

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x) = f(-1)$$

$$\Rightarrow -a + b = b - 3$$

$$\Rightarrow \underline{a = 3}$$

$$f'(x) = \begin{cases} a, & x > -1 \\ 2bx, & x \leq -1 \end{cases}$$

$$\Rightarrow a = -2b = 3 \Rightarrow \underline{b = -\frac{3}{2}}$$

Figure 9: Solution to Section 3.3, problem 70

$$\begin{aligned}
 4. \quad & \frac{d}{dx} \begin{vmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{vmatrix} \\
 &= \frac{d}{dx} (f_{11}(x)f_{22}(x) - f_{12}(x)f_{21}(x)) \\
 &= \underline{f_{11}'(x)f_{22}(x) + f_{11}(x)f_{22}'(x)} - \underline{f_{12}'(x)f_{21}(x) + f_{12}(x)f_{21}'(x)} \\
 &= \begin{vmatrix} f_{11}'(x) & f_{12}'(x) \\ f_{21}(x) & f_{22}(x) \end{vmatrix} + \begin{vmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x)' & f_{22}'(x) \end{vmatrix} \\
 &= \begin{vmatrix} f_{11}'(x) & f_{12}(x) \\ f_{21}'(x) & f_{22}(x) \end{vmatrix} + \begin{vmatrix} f_{11}(x) & f_{12}'(x) \\ f_{21}(x) & f_{22}'(x) \end{vmatrix}
 \end{aligned}$$

Figure 10: Solution to Homework 03, problem 5

$$\begin{aligned}
 6. \quad & \frac{d^2}{dx^2} (u(x)v(x)) = u''(x)v(x) + 2u'(x)v'(x) + u(x)v''(x) \\
 & \frac{d^3}{dx^3} (u(x)v(x)) = u'''(x)v(x) + 3u''(x)v'(x) + 3u'(x)v''(x) + u(x)v'''(x) \\
 & \frac{d^n}{dx^n} (u(x)v(x)) = \sum_{i=0}^n \binom{n}{i} \frac{d^i}{dx^i} u(x) \frac{d^{n-i}}{dx^{n-i}} v(x) \quad \binom{n}{i} = \frac{n!}{(n-i)! i!}
 \end{aligned}$$

Figure 11: Solution to Homework 03, problem 6