

Brief solutions to selected problems in homework 02

1. Section 2.4: Solutions, common mistakes and corrections:

2. Problem 25

$$\lim_{x \rightarrow 0} \frac{\sin(1 - \cos x)}{x}$$

~~$\lim_{x \rightarrow 0} \frac{\sin^2(\frac{x}{2})}{\frac{x}{2}}$~~

$$= \lim_{x \rightarrow 0} \frac{\sin(1 - \cos x)}{1 - \cos x} \cdot \frac{1 - \cos x}{x}$$

$$= 1 \times \left(\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \right) = 1 \times 0 = 0$$

~~$= \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \cdot \sin \frac{x}{2} = \lim_{x \rightarrow 0} \sin \frac{x}{2} = 0$~~

$1 - \cos x = 2 \sin^2 \frac{x}{2}$
 $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{x/2} \sin \frac{x}{2} = 1 \times 0 = 0$

Figure 1: Chapter 2: Additional and advanced Exercise, problem 25

2. Section 2.5: Solutions, common mistakes and corrections:

2.5.64. f, g conti. at $x=0$, but $f \circ g$ disconti. at $x=0$.
 $f(x), g(x)$ is continuous at $x=0$, Thm. g conti. at c , f conti. at $g(c)$
 $\rightarrow f \circ g$ conti. at c .
 $g(0) \neq 0$ is possible,
 $"f(x)$ at $g(0)$ is continuous" is not mentioned,
 so $f(g(x)) = (f \circ g)(x)$ is possibly not continuous.
 $g(x) = x+1 \Rightarrow$ conti. at $x=0$.
 $f(x) = \begin{cases} 1 & \text{if } x=1 \\ 0 & \text{if } x \neq 1 \end{cases} \Rightarrow f$ conti. at $x=0$.
 $(f \circ g)(x) = f(x+1) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x \neq 0 \end{cases} \Rightarrow$ disconti. at $x=0$.

Figure 2: Section 2.5, problem 64. Ans: no. A counter example

Sec. 2.5 #64

Take $g(x) = x + 1$

$$f(x) = \frac{1}{x-1}$$

$$f \circ g(x) = f(g(x)) = \frac{1}{g(x)-1} = \frac{1}{x+1-1} = \frac{1}{x}$$

Figure 3: Section 2.5, problem 64. Ans: no. Another counter example

2.5. 67.

設 $g(x) = f(x) - x$

$\because g(0) = f(0) - 0 \geq 0$ 且 $g(1) = f(1) - 1 \leq 0$

若 $g(0) = 0$ 則 $c = 0$, 若 $g(1) = 0$ 則 $c = 1$

若兩者皆不成立 由中間值定理 ($\because g$ conti. on $[0, 1]$)

$\exists c \in (0, 1)$ 使得 $g(c) = 0 \Leftrightarrow f(c) = c$

Figure 4: Section 2.5, problem 67

Given $\varepsilon = \frac{|f(c)|}{2} > 0$, there exists $\delta > 0$ such that

$0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$ Because $f(x)$ is conti.
at c , $\lim_{x \rightarrow c} f(x) = f(c)$

$-\frac{|f(c)|}{2} < f(x) - f(c) < \frac{|f(c)|}{2}$ ① if $f(c) > 0$:

$-\frac{|f(c)|}{2} + f(c) < f(x) < \frac{|f(c)|}{2} + f(c)$ $0 < -\frac{f(c)}{2} + f(c) < f(x)$ for $x \in (c - \delta, c + \delta)$

② if $f(c) < 0$:

$f(x) < f(c) + \frac{|f(c)|}{2} = \frac{f(c)}{2} < 0$

for $x \in (c - \delta, c + \delta)$

Figure 5: Solution to Section 2.5, problem 68

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suppose $f(c) > 0$. by continuity, for any ε there exists a

such that $0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

$0 < |x - c| < \delta \Rightarrow -\varepsilon < f(x) - f(c) < \varepsilon$

take $\varepsilon = \frac{f(c)}{2} \Rightarrow \frac{1}{2}f(c) < f(x) < \frac{3}{2}f(c)$ if $0 < |x - c| < \delta$

$\therefore f(x)$ have the same ^{and corresponding δ} sign as $f(c)$ $f(c) < 0$
take $\varepsilon = -f(c)/2$

Figure 6: Common mistakes to Section 2.5, problem 68