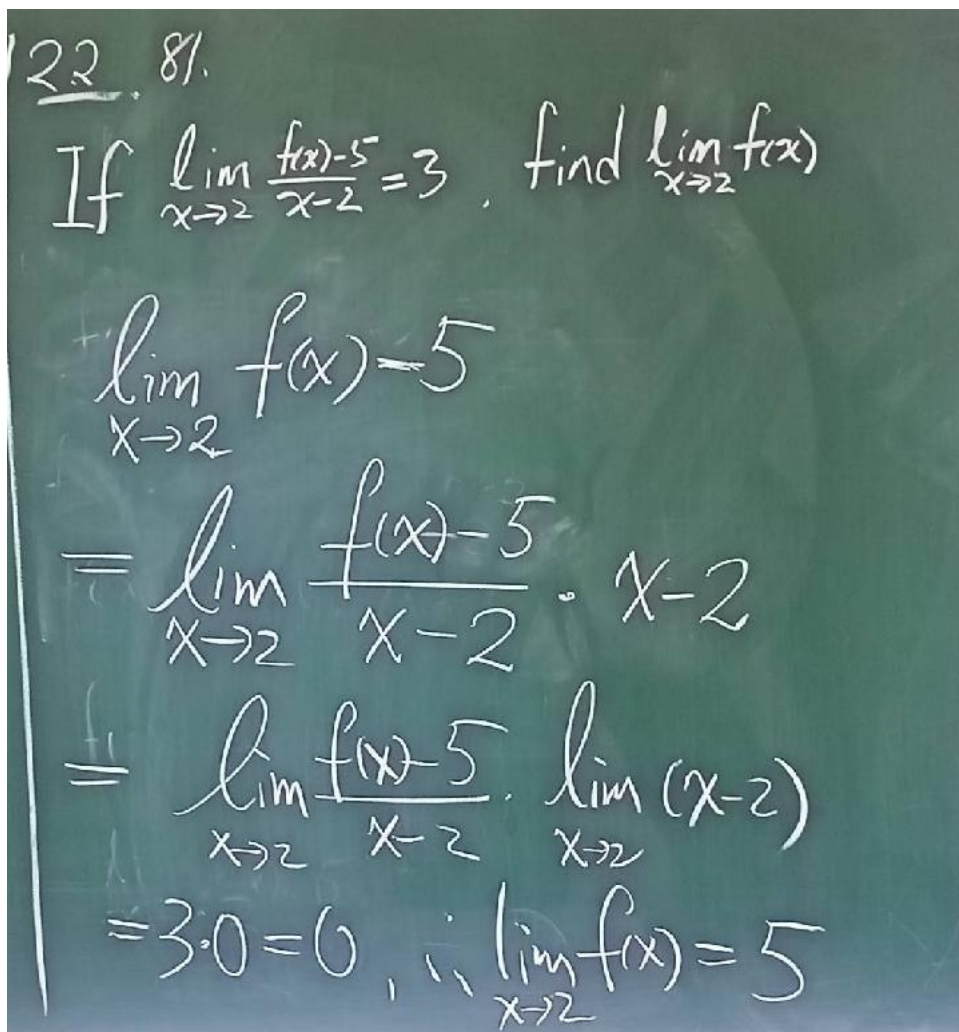


## Brief solutions to selected problems in homework 01

1. Section 2.2: Solutions, common mistakes and corrections:



2.2. 81.  
If  $\lim_{x \rightarrow 2} \frac{f(x)-5}{x-2} = 3$ , find  $\lim_{x \rightarrow 2} f(x)$   
$$\lim_{x \rightarrow 2} f(x) - 5$$
$$= \lim_{x \rightarrow 2} \frac{f(x)-5}{x-2} \cdot (x-2)$$
$$= \lim_{x \rightarrow 2} \frac{f(x)-5}{x-2} \cdot \lim_{x \rightarrow 2} (x-2)$$
$$= 3 \cdot 0 = 0, \therefore \lim_{x \rightarrow 2} f(x) = 5$$

Figure 1: Section 2.2, problem 81

**Remark:** The idea: Since "the limit of the ratio" is nonzero and "the limit of the denominator" is zero, it follows that "the limit of the numerator" must be zero, otherwise the ratio will diverge. The idea can be supported by the Limit Laws as detailed above.



s2.3 p43

$$\lim_{x \rightarrow 1} \frac{1}{x} = 1$$

$$0 < |x-1| < \delta \Rightarrow \left| \frac{1}{x} - 1 \right| < \varepsilon$$

$$-\varepsilon < \frac{1}{x} - 1 < \varepsilon \quad \Leftrightarrow \quad \frac{\varepsilon}{-\varepsilon+1} > x-1 > \frac{-\varepsilon}{\varepsilon+1} \quad \left( \begin{array}{l} \text{Assume} \\ 0 < \varepsilon < 1 \end{array} \right)$$

$$\Leftrightarrow -\varepsilon+1 < \frac{1}{x} < \varepsilon+1$$

$$\Leftrightarrow \frac{1}{\varepsilon+1} < x < \frac{1}{-\varepsilon+1}$$

$$\Leftrightarrow \frac{1}{\varepsilon+1} < x-1 < \frac{1}{-\varepsilon+1} - 1$$

take  $\delta = \min \left( \frac{\varepsilon}{-\varepsilon+1}, \frac{\varepsilon}{\varepsilon+1} \right)$

If  $\varepsilon \geq 1$ , take  $\delta = \delta(\frac{1}{2})$  from  $\varepsilon = \frac{1}{2}$

$$0 < |x-1| < \delta(\frac{1}{2}) \Rightarrow \left| \frac{1}{x} - 1 \right| < \frac{1}{2} < \varepsilon$$

Figure 4: Section 2.3, problem 43.

2.3 (43)  $\lim_{x \rightarrow 1} \frac{1}{x} = 1$

For any  $\varepsilon > 0$   $1 > \varepsilon > 0$

$$\left| \frac{1}{x} - 1 \right| < \varepsilon$$

$$\Leftrightarrow -\varepsilon < \frac{1}{x} - 1 < \varepsilon$$

$$\Leftrightarrow 1-\varepsilon < \frac{1}{x} < 1+\varepsilon$$

$$\Leftrightarrow \frac{1}{1+\varepsilon} > x > \frac{1}{1-\varepsilon}$$

$$\Leftrightarrow \frac{1}{1-\varepsilon} - 1 > x-1 > \frac{1}{1+\varepsilon} - 1$$

$$\Rightarrow \delta = \min \left\{ \frac{1}{1+\varepsilon} - 1, \frac{1}{1-\varepsilon} - 1 \right\}$$

take  $\delta = \frac{1}{1+\varepsilon} - 1$

$$= \frac{1-1-\varepsilon}{1+\varepsilon}$$

$$= \frac{-\varepsilon}{1+\varepsilon}$$

Figure 5: Section 2.3, problem 43: mistake 1

43.

$$0 < |x-1| < \delta$$

$$\Rightarrow -\delta < x-1 < \delta$$

$$\Rightarrow -\delta+1 < x < \delta+1$$

$$|\frac{1}{x}-1| < \epsilon \quad (A)$$

$$\Leftrightarrow -\epsilon < \frac{1}{x}-1 < \epsilon$$

$$\Leftrightarrow -\epsilon+1 < \frac{1}{x} < \epsilon+1 \quad \text{assume } 0 < \epsilon < 1$$

$$\Leftrightarrow \frac{1}{1+\epsilon} < x < \frac{1}{1-\epsilon} \quad (B)$$

$$\begin{cases} \delta+1 \leq \frac{1}{1-\epsilon} \\ -\delta+1 \geq \frac{1}{1+\epsilon} \end{cases} \Rightarrow \begin{cases} \delta \leq \frac{1}{1-\epsilon}-1 \\ \delta \leq 1-\frac{1}{1+\epsilon} \end{cases}$$

$$\text{take } \delta = \min \left\{ \frac{1}{1-\epsilon}-1, 1-\frac{1}{1+\epsilon} \right\}$$

因為要說明所選取的delta可以從(B)推到(A), 反向箭頭是必要的

Figure 6: Section 2.3, problem 43: mistake 2

43.  $\lim_{x \rightarrow 1} \frac{1}{x} = 1$

for any  $\epsilon > 0$ , there exist  $\delta > 0$

$$0 < |x-1| < \delta \Rightarrow \left| \frac{1}{x} - 1 \right| < \epsilon$$

$$\Leftrightarrow -\epsilon < \frac{1}{x} - 1 < \epsilon$$

$$\Leftrightarrow 1-\epsilon < \frac{1}{x} < 1+\epsilon$$

$$\Leftrightarrow \frac{1}{1+\epsilon} > x > \frac{1}{1-\epsilon}$$

$$\Leftrightarrow \frac{1-(1+\epsilon)}{1-\epsilon} > x-1 > \frac{1-(1-\epsilon)}{1+\epsilon}$$

$$\Leftrightarrow \frac{-\epsilon}{1-\epsilon} < x-1 < \frac{\epsilon}{1+\epsilon}$$

$$\text{take } \delta = \min \left( \frac{\epsilon}{1-\epsilon}, \frac{\epsilon}{1+\epsilon} \right)$$

$$\Leftrightarrow 0 < |x-1| < \delta \Rightarrow \left| \frac{1}{x} - 1 \right| < \epsilon$$

Figure 7: Section 2.3, problem 43: mistake 3



4. Problem 4: False. Counter example:

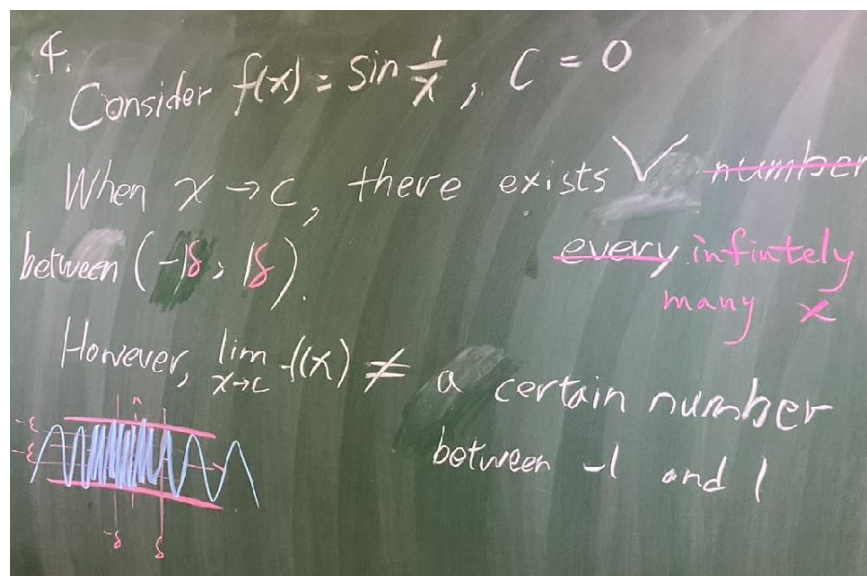


Figure 8: Homework 01, problem 4, a counter example

5. Problem 5:

2. If  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} g(x) = M$

For any  $\epsilon > 0$ , there exists  $\delta_1 > 0$  and  $\delta_2 > 0$ , such that

$$0 < |x - c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{6}$$

$$0 < |x - c| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{6}$$

$$\Leftrightarrow \begin{cases} -\frac{2}{3}\epsilon < 4f(x) - 4L < \frac{2}{3}\epsilon \\ -\frac{1}{3}\epsilon < -2g(x) + 2M < \frac{1}{3}\epsilon \end{cases}$$

take  $\delta = \min(\delta_1, \delta_2)$

$$0 < |x - c| < \delta \Rightarrow -\epsilon < 4f(x) - 2g(x) - (4L - 2M) < \epsilon$$

$$\Leftrightarrow |4f(x) - 2g(x) - (4L - 2M)| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow c} (4f(x) - 2g(x)) = 4L - 2M$$

Figure 9: Homework 01, problem 5