

Volume and Surface Area of Revolution (V04)

Notation:

a : axial variable. r : radial variable. (a, r) could be (x, y) or (y, x) .

The generating region:

$$\begin{aligned}\mathcal{R} &= \{(a, r), \alpha < a < \beta, 0 \leq r_1(a) < r < r_2(a)\} \\ &= \{(a, r), 0 \leq \gamma < r < \delta, a_1(r) < a < a_2(r)\}\end{aligned}$$

Volume of revolution generated by region $\mathcal{R}$:

Method of disks ($r_1 = 0$) or washers ($r_1 > 0$):

$$V = \int_{\alpha}^{\beta} \pi(r_2^2(a) - r_1^2(a)) da$$

Method of cylindrical shells:

$$V = \int_{\gamma}^{\delta} 2\pi r(a_2(r) - a_1(r)) dr$$

Arclength of curve $\mathcal{C}$ and area of surface of revolution generated by $\mathcal{C}$:

General formula: $L = \int_{*}^{*} ds$, $ds = \sqrt{dx^2 + dy^2}$.

$\mathcal{S} = \int_{*}^{*} 2\pi r ds$, where $r = x$ if rotated around y -axis, $r = y$ if rotated around x -axis.

Case 1: If $\mathcal{C} = \{y = f(x) \geq 0, a \leq x \leq b\}$, then $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + (f'(x))^2} dx,$$

Surface area:

Rotate $\mathcal{C}$ around x -axis:

$$S_{x\text{-axis}} = \int_a^b 2\pi r ds = \int_a^b 2\pi y ds = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

Rotate $\mathcal{C}$ around y -axis:

$$S_{y\text{-axis}} = \int_a^b 2\pi r ds = \int_a^b 2\pi x ds = \int_a^b 2\pi x \sqrt{1 + (f'(x))^2} dx$$

Case 2: If $\mathcal{C} = \{x = g(y) \geq 0, c \leq y \leq d\}$, then $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$.

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d \sqrt{1 + (g'(y))^2} dy,$$

Surface area

Rotate $\mathcal{C}$ around x -axis:

$$S_{x\text{-axis}} = \int_c^d 2\pi r ds = \int_c^d 2\pi y ds = \int_c^d 2\pi y \sqrt{1 + (g'(y))^2} dy$$

Rotate $\mathcal{C}$ around y -axis:

$$S_{y\text{-axis}} = \int_c^d 2\pi r ds = \int_c^d 2\pi x ds = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} dy$$

Case 3: If $\mathcal{C} = \{x = X(t), y = Y(t), \alpha \leq t \leq \beta\}$, then $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_\alpha^\beta \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)} dt$$

$$S_{x\text{-axis}} = \int_\alpha^\beta 2\pi r ds = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_\alpha^\beta 2\pi Y(t) \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)} dt$$

$$S_{y\text{-axis}} = \int_\alpha^\beta 2\pi r ds = \int_\alpha^\beta 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_\alpha^\beta 2\pi X(t) \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)} dt$$