

Thm 2. If  $f, g$  are cont.  
on  $[a, \infty)$  and  $0 \leq f \leq g$

Then

$$(1) \int_a^{\infty} g(x) dx < \infty \Rightarrow \int_a^{\infty} f(x) dx < \infty$$

(conv.) (conv.)

$$(2) \int_a^{\infty} f(x) dx = \infty \Rightarrow \int_a^{\infty} g(x) dx = \infty$$

Similarly for  $\lim_{x \rightarrow -\infty}$  and type II.

Eg 1  $\int_0^{\infty} e^{-x^2} dx$  conv?

Sol Compare with  $\int_0^{\infty} e^{-x} dx$

$$e^{-x^2} \leq e^{-x} \quad \text{on } x \geq 1$$

$$\int_1^{\infty} e^{-x^2} dx \leq \int_1^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} \int_1^b e^{-x} dx$$

$$= \lim_{b \rightarrow \infty} -e^{-x} \Big|_1^b = \lim_{b \rightarrow \infty} (e^{-1} - e^{-b}) < \infty$$

$$\Rightarrow \int_1^{\infty} e^{-x^2} dx < \infty \Rightarrow \int_0^{\infty} e^{-x^2} dx < \infty$$

Thm 3 If  $f, g$  are cont.  
on  $[a, \infty)$  and  $f > 0, g > 0$ .

$$\text{and } \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$$

(1)  $0 < L < \infty$

$$\int_a^{\infty} f(x) dx < \infty \iff \int_a^{\infty} g(x) dx < \infty$$

(2)  $L = 0$

" $\Leftarrow$ "

(3)  $L = \infty$

" $\Rightarrow$ "

Similarly for  $(-\infty, b]$  and type II

Eg 2. Check convergence

	$\int_a^b f(x) dx$	$g(x)$	Thm	conv/div
(a)	$\int_1^{\infty} \frac{\sin^2 x}{x^2} dx$	$\frac{1}{x^2}$	2-(1)	conv
(b)	$\int_1^{\infty} \frac{1}{\sqrt{x^2+1}} dx$	$\frac{1}{x}$	3-(1)	div.
(c)	$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{x}} dx$	$\frac{1}{\sqrt{x}}$	$\begin{cases} 2-(1) \\ 3-(1) \\ x \rightarrow 0^+ \end{cases}$	conv
(d)	$\int_1^{\infty} \frac{1-e^{-x}}{x} dx$	$\frac{1}{x}$	3-(1)	div.

$$\text{Ex 3. } \int_1^{\infty} \frac{1}{x(x^2-1)^{\frac{1}{3}}} dx \text{ conv?}$$

$$\text{Sol} = \int_1^2 + \int_2^{\infty}$$

$$f(x) = \frac{1}{x(x^2-1)^{\frac{1}{3}}}$$

$$\textcircled{a} \int_1^2: \text{ take } g(x) = \frac{1}{(x-1)^{\frac{1}{3}}}$$

$$\lim_{x \rightarrow 1^+} \frac{f(x)}{g(x)} = 2^{\frac{1}{3}}$$

$$\int_1^2 g(x) dx < \infty \xRightarrow{\text{Thm 3-1}} \int_1^2 f(x) dx < \infty$$

$(p = \frac{1}{3})$

$$\textcircled{b} \int_2^{\infty} : \text{take } g(x) = \frac{1}{x^{1+\frac{2}{3}}}$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{(x^2)^{\frac{1}{3}}}{(x^2-1)^{\frac{1}{3}}} = 1$$

$$\int_2^{\infty} g(x) dx < \infty \quad (p = \frac{5}{3})$$

$$\text{Thm 3-(1)} \Rightarrow \int_2^{\infty} f(x) dx < \infty$$

$$\textcircled{a}, \textcircled{b} \Rightarrow \int_1^{\infty} f(x) dx < \infty$$

$$\text{Ex 4 } \int_e^{\infty} \underbrace{x^{-2}}_{f(x)} \ln x \, dx \text{ conv?}$$

Take  $g(x) = x^{-2} \rightarrow$  No conclusion

Take  $g(x) = x^{-1.5}$  instead  
( $x^{-1.9}, x^{-1.99}, \dots$  etc)

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{\ln x}{x^{0.5}} \\ &= (\text{l'Hopital}) = 0 \end{aligned}$$

Thm 3 - (2)

$$\Rightarrow \int_e^{\infty} f(x) \ln x \, dx < \infty$$

$$\text{Eg 5 } \int_e^{\infty} x^{-0.5} \ln x \, dx$$

Sol take  $g(x) = x^{-0.5}$

$$f(x) > g(x), \int_e^{\infty} g(x) \, dx = \infty$$

$$\Rightarrow \int_e^{\infty} f(x) \, dx = \infty$$

$$\text{Eg 6 } \int_0^{1/e} x^{-0.5} (-\ln x) \, dx$$

Sol take  $g(x) = x^{-\frac{2}{3}}$ ,  $\int_0^{1/e} x^{-\frac{2}{3}} \, dx < \infty$

$$\lim_{x \rightarrow 0^+} \frac{x^{-0.5} (-\ln x)}{x^{-\frac{2}{3}}} = \lim_{x \rightarrow 0^+} \left( -\frac{\ln x}{x^{\frac{1}{6}}} \right) \stackrel{\text{L'Hopital}}{=} 0$$

Thm 3-(2)  $\Rightarrow \int_0^{1/e} f(x) \, dx$  conv.