

Improper Integrals (瑕積分)

Recall (proper) integrals:

If $f(x)$ is cont. on $[a, b]$

$$\Rightarrow |f(x)| \leq M \text{ on } [a, b]$$

$$\Rightarrow \int_a^b f(x) dx \text{ exists}$$

If either $(b-a)$ or $\max|f|$
becomes $\infty \Rightarrow$ Improper integrals

Def: (type I)

(1) If f is cont. on $[a, \infty)$.

$$\int_a^{\infty} f(x) dx \stackrel{\text{def}}{=} \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

(2) If f is cont. on $(-\infty, b]$

$$\int_{-\infty}^b f(x) dx \stackrel{\text{def}}{=} \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

(3) If f is cont. on $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} f(x) dx \stackrel{\text{def}}{=} \int_{-\infty}^c + \int_c^{\infty} \quad \text{for any } c \in \mathbb{R}$$

Limit exists $\Leftrightarrow$ Improper
and finite integral
converges
Otherwise $\Leftrightarrow$ diverges

In case (3), $\int_{-\infty}^{\infty} f(x) dx$

Converges if both
 $\int_{-\infty}^c$ and $\int_c^{\infty}$ exist and
are finite (converge)

If either $\int_{-\infty}^c$ or $\int_c^{\infty}$ diverges
 $\Leftrightarrow \int_{-\infty}^{\infty}$ diverges

Remark: If $f \geq 0$
then $\int_a^\infty$, $\int_{-\infty}^b$ or $\int_{-\infty}^\infty$
either converges or
diverges to $+\infty$.

We write

$$\int_a^\infty f(x) dx = \infty$$

to mean it diverges
to $+\infty$

Eg 1 $\int_1^{\infty} \frac{\ln x}{x^2} dx = ?$

Sol. $\int_1^b \frac{\ln x}{x^2} dx = \int_1^b \ln x d\left(\frac{-1}{x}\right)$

$$= (\ln x) \left(\frac{-1}{x}\right) \Big|_1^b - \int_1^b \frac{-1}{x} d \ln x$$

$$= \frac{-1}{b} \ln b + \int_1^b \frac{1}{x} \left(\frac{dx}{x}\right)$$

$$= \frac{-1}{b} \ln b - \left(\frac{1}{b} - 1\right)$$

$$\int_1^{\infty} = \lim_{b \rightarrow +\infty} \int_1^b = 1$$

Ex 2: $\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx$ converges?

Sol: Can we do

$$\lim_{b \rightarrow \infty} \int_{-b}^b \frac{x}{1+x^2} dx ? \quad \underline{\text{No!}}$$

$$(\quad = \lim_{b \rightarrow \infty} 0 = 0 \quad)$$

Correct computation:

Find $\int_0^{\infty}$ and $\int_{-\infty}^0$ separately.

$$\begin{aligned} \lim_{b \rightarrow \infty} \int_0^b \frac{x}{1+x^2} dx &= \lim_{b \rightarrow \infty} \frac{1}{2} \int_0^b \frac{d(x^2)}{1+x^2} \\ &= \lim_{b \rightarrow \infty} \frac{1}{2} \int_0^b \frac{d(1+x^2)}{1+x^2} = \lim_{b \rightarrow \infty} \frac{\ln(1+b^2)}{2} = \infty \end{aligned}$$

Since $\int_0^{\infty}$ diverges

$\Rightarrow \int_{-\infty}^{\infty}$ diverges (by definition)

Remark:

$$\lim_{b \rightarrow \infty} \int_{-b}^{zb} \frac{x}{1+x^2} dx$$

$$= \lim_{b \rightarrow \infty} \frac{1}{2} \ln \left(\frac{1+(zb)^2}{1+b^2} \right) \quad \leftarrow$$

$$= \ln 2$$

This example explains why we must define $\int_{-\infty}^{\infty}$ as $\int_{-\infty}^c + \int_c^{\infty}$

Ex 3 $\int_1^{\infty} \frac{1}{x^p} dx, p > 0$

Ans $\begin{cases} p > 1 & \text{converges} \\ 0 < p \leq 1 & \text{diverges} \end{cases} \left(\frac{1}{x^p} \right)_A^t$

Sol

$$p \neq 1 \quad \int_1^{\infty} x^{-p} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-p} dx$$
$$= \lim_{b \rightarrow \infty} \left(\frac{1}{-p+1} x^{-p+1} \Big|_1^b \right)$$

$$p = 1 \quad \int_1^{\infty} x^{-1} dx = \lim_{b \rightarrow \infty} (\ln x \Big|_1^b) = \infty$$

Def: (type II)

(1) If $f(x)$ is cont. on $(a, b]$

$$\int_a^b f(x) dx \stackrel{\text{def}}{=} \lim_{c \rightarrow a^+} \int_c^b f(x) dx$$

(2) If $f(x)$ is cont. on $[a, b)$

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$$

(3) If $f(x)$ is cont. on (a, b)

$$\int_a^b f(x) dx \stackrel{\text{def}}{=} \int_a^c f(x) dx + \int_c^b f(x) dx$$

for any $c \in (a, b)$ (any $c \in (a, b)$ will do)

Eg4: $\int_0^1 \frac{1}{x^p} dx$, $p > 0$

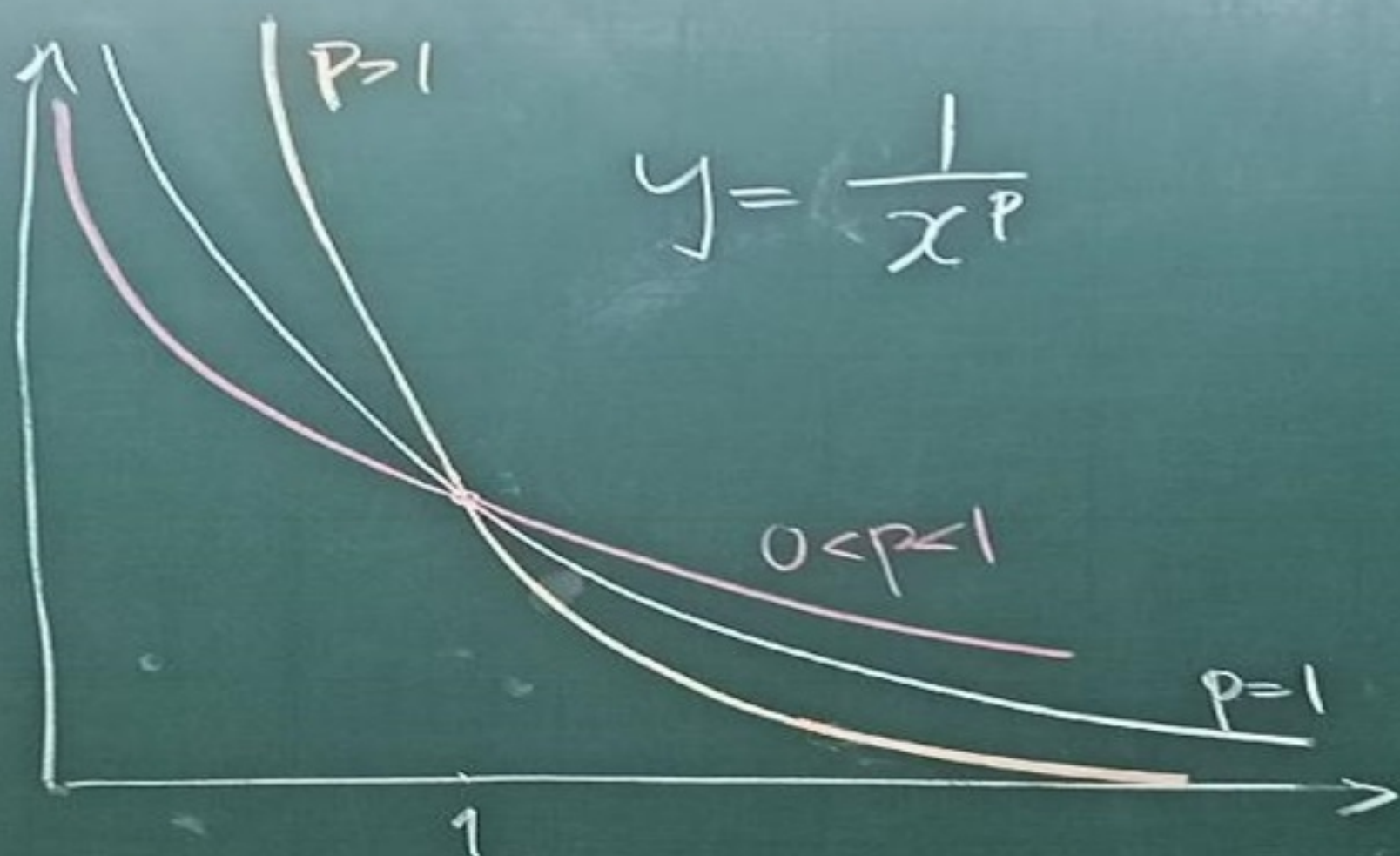
Ans: $p \geq 1$: diverges
 $0 < p < 1$: converges ($\frac{1}{A}$)

$$p \neq 1: \int_0^1 x^{-p} dx = \lim_{c \rightarrow 0^+} \int_c^1 x^{-p} dx$$

$$= \lim_{c \rightarrow 0^+} \left(\frac{1}{-p+1} x^{-p+1} \Big|_c^1 \right)$$

$$p=1: \int_0^1 x^{-1} dx = \lim_{c \rightarrow 0^+} \left(\ln x \Big|_c^1 \right)$$

$$= +\infty$$



Remark: The same conclusion applies to $\int_c^{c+1} \frac{1}{(x-c)^p} dx$ and $\left(\int_{c-1}^c \frac{1}{(x-c)^p} dx \right)$ by the change of variable $y = x - c$ ($c - x$).

Remark : Same conclusion

$$\begin{aligned} &\text{holds for } \int_2^{\infty} \frac{1}{x^p} dx \\ &\text{or } \int_{-\infty}^{-1} \frac{1}{|x|^p} dx \\ &\text{or } \int_{C+1}^{\infty} \frac{1}{(x-C)^p} dx, \text{ etc.} \end{aligned} \left. \vphantom{\int_2^{\infty} \frac{1}{x^p} dx} \right\} \text{Type I}$$

$$\begin{aligned} &\text{and } \int_0^2 \frac{1}{x^p} dx, \\ &\text{or } \int_{-1}^0 \frac{1}{|x|^p} dx, \text{ etc.} \end{aligned} \left. \vphantom{\int_0^2 \frac{1}{x^p} dx} \right\} \text{Type II}$$

Eg 5. $\int_0^1 \frac{1}{1-x} dx$ converge?

Ans: This is $p=1$ (type II)
it diverges

$$\text{Sol} = \lim_{C \rightarrow 1^-} \int_0^C \frac{1}{1-x} dx$$

$$\text{let } y = 1-x, \quad dx = -dy$$

$$= \lim_{C \rightarrow 1^-} - \int_{y=1}^{1-C} \frac{dy}{y} = \lim_{C \rightarrow 1^-} - \ln y \Big|_1^{1-C}$$

$$= \lim_{C \rightarrow 1^-} -\ln(1-C) = +\infty \text{ (diverges)}$$

Ex 6: $\int_0^{\infty} \frac{1}{x^p} dx$ converges?
($p > 0$)

Ans.: Type I at $[1, \infty)$
+ Type II at $(0, 1]$

$$\int_0^{\infty} = \int_0^1 + \int_1^{\infty}$$

$$= \begin{cases} \text{finite} & + \infty & 0 < p < 1 \\ +\infty & + \infty & p = 1 \\ +\infty & + \text{finite} & p > 1 \end{cases}$$

= divergent (at least one of $\int_0^1$ or $\int_1^{\infty}$ diverges)

Ex 7. $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$ converges?

Sol: Check $\int_0^1$ and $\int_1^3$
separately. (both type II)

From previous remark,
both $\int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}}$ and $\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$

converge by the change
of variable. $y=1-x$ or $y=x-1$

and $p=\frac{2}{3}$ (type II) $\Rightarrow \int_0^3$ converges