Improper Integrals
(瑕積分)

Recall (proper) integrals:

If $f(x)$ is cont. on $[a, b]$

$\Longrightarrow |f(x)| \leq M$ on $[a, b]$

$\Longrightarrow \int_a^b f(x)\,dx$ exists

If either $(b-a)$ or $\max|f|$ becomes $\infty \Longrightarrow$ Improper integrals

Def: (type I)

(1) If f is cont. on $[a,\infty)$.

$$\int_a^{\infty} f(x)\,dx \overset{\text{def}}{=\!=} \lim_{b\to\infty} \int_a^b f(x)\,dx$$

(2) If f is cont. on $(-\infty, b]$

$$\int_{-\infty}^{b} f(x)\,dx \overset{\text{def}}{=\!=} \lim_{a\to -\infty} \int_a^b f(x)\,dx$$

(3) If f is cont. on $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} f(x)\,dx \overset{\text{def}}{=\!=} \int_{-\infty}^{c} + \int_{c}^{\infty} \quad \text{for any } c \in \mathbb{R}$$

Limit exists and finite $\Longleftrightarrow$ Improper integral converges

Otherwise $\Longleftrightarrow$ diverges

In case (3). $\int_{-\infty}^{\infty} f(x)\,dx$ converges if both $\int_{-\infty}^{c}$ and $\int_{c}^{\infty}$ exist and are finite (converge)

If either $\int_{-\infty}^{c}$ or $\int_{c}^{\infty}$ diverges $\Leftrightarrow \int_{-\infty}^{\infty}$ diverges

Remark: If $f \geq 0$

then $\int_a^\infty$, $\int_{-\infty}^b$ or $\int_{-\infty}^\infty$

either converges or

diverges to $+\infty$.

We write

$$\int_a^\infty f(x)\,dx = \infty$$

to mean it diverges

to $+\infty$

Eg1 $\int_1^\infty \frac{\ln x}{x^2}\,dx = ?$

<u>Sol</u>: $\int_1^b \frac{\ln x}{x^2}\,dx = \int_1^b \ln x \, d\left(\frac{-1}{x}\right)$

$= (\ln x)\left(\frac{-1}{x}\right)\Big|_1^b - \int_1^b \frac{-1}{x}\, d\ln x$

$= \frac{-1}{b}\ln b + \int_1^b \frac{1}{x}\left(\frac{dx}{x}\right)$

$= \frac{-1}{b}\ln b - \left(\frac{1}{b} - 1\right)$

$\int_1^\infty = \lim_{b\to+\infty} \int_1^b = 1$

Eg 2: $\displaystyle\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$ converges?

<u>Sol</u>: Can we do

$$\lim_{b\to\infty} \int_{-b}^{b} \frac{x}{1+x^2}\,dx \ ? \quad \underline{\text{No}}!$$

$$\left(= \lim_{b\to\infty} 0 = 0\right)$$

Correct computation:

Find $\int_0^{\infty}$ and $\int_{-\infty}^{0}$ separately.

$$\lim_{b\to\infty}\int_0^b \frac{x}{1+x^2}\,dx = \lim_{b\to\infty}\frac{1}{2}\int_0^b \frac{d(x^2)}{1+x^2}$$

$$= \lim_{b\to\infty}\frac{1}{2}\int_0^b \frac{d(1+x^2)}{1+x^2} = \lim_{b\to\infty}\frac{\ln(1+b^2)}{2} = \infty$$

Since $\int_0^\infty$ diverges

$\Longrightarrow \int_{-\infty}^{\infty}$ diverges (by definition)

Remark:

$$\lim_{b\to\infty} \int_{-b}^{2b} \frac{x}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2} \ln\left(\frac{1+(2b)^2}{1+b^2}\right) \leftarrow$$

$$= \ln 2$$

This example explains why we must define $\int_{-\infty}^{\infty}$ as $\int_{-\infty}^{c} + \int_{c}^{\infty}$

Eg 3 $\int_1^\infty \frac{1}{x^p}dx$, $p>0$

Ans $\begin{cases} p>1 & \text{converges} \\ 0<p\le 1 & \text{diverges} \end{cases}$ ($\frac{2t}{A}$)

Sol

$p\neq 1$ $\int_1^\infty x^{-p}dx = \lim_{b\to\infty}\int_1^b x^{-p}dx$

$= \lim_{b\to\infty}\left(\frac{1}{-p+1}x^{-p+1}\Big|_1^b\right)$

$p=1$ $\int_1^\infty x^{-1}dx = \lim_{b\to\infty}\left(\ln x\Big|_1^b\right) = \infty$

Def: (type II)

(1) If $f(x)$ is cont. on $(a, b]$

$$\int_a^b f(x)\,dx \overset{\text{def}}{=\!=} \lim_{c \to a^+} \int_c^b f(x)\,dx$$

(2) If $f(x)$ is cont. on $[a, b)$

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

(3) If $f(x)$ is cont. on (a, b)

$$\int_a^b f(x)\,dx \overset{\text{def}}{=\!=} \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

for any $c \in (a, b)$ (any $c \in (a, b)$ will do)

Eg4: $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$, $p>0$

<u>Ans</u>: $p \geq 1$: diverges

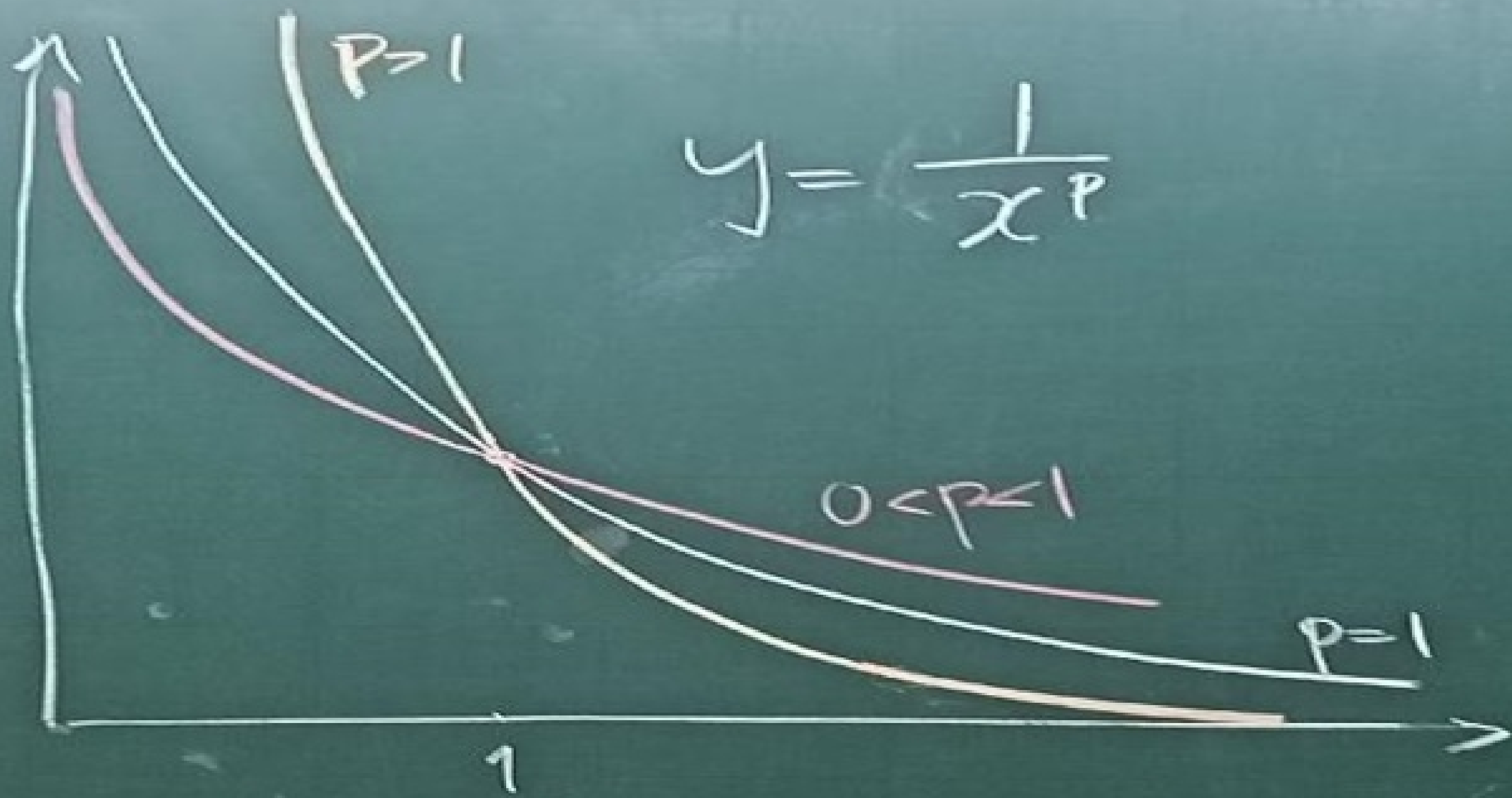
$0<p<1$: converges ([illegible])

$p \neq 1$: $\displaystyle\int_0^1 x^{-p}\,dx = \lim_{c\to 0^+} \int_c^1 x^{-p}\,dx$

$\displaystyle = \lim_{c\to 0^+}\left(\frac{1}{-p+1}\, x^{-p+1}\Big|_c^1\right)$

$p=1$: $\displaystyle\int_0^1 x^{-1}\,dx = \lim_{c\to 0^+}\left(\ln x\Big|_c^1\right)$

$= +\infty$



Remark: The same conclusion applies to $\int_c^{c+1} \frac{1}{(x-c)^p}\,dx$ and $\left(\int_{c-1}^{c} \frac{1}{|x-c|^p}\,dx\right)$ by the change of variable $y=x-c$ $(c-x)$

<u>Remark</u>: Same conclusion

holds for

$$\left.\begin{array}{l} \displaystyle\int_{2}^{\infty} \frac{1}{x^{p}}\,dx \\ \text{or } \displaystyle\int_{-\infty}^{-1} \frac{1}{|x|^{p}}\,dx \\ \text{or } \displaystyle\int_{c+1}^{\infty} \frac{1}{(x-c)^{p}}\,dx, \text{ etc} \end{array}\right\} \text{Type I}$$

$$\left.\begin{array}{l} \text{and } \displaystyle\int_{0}^{2} \frac{1}{x^{p}}\,dx, \\ \text{or } \displaystyle\int_{-1}^{0} \frac{1}{|x|^{p}}\,dx. \text{ etc} \end{array}\right\} \text{Type II}$$

Eg 5. $\int_0^1 \frac{1}{1-x}\,dx$ converge ?

<u>Ans</u>: This is $p=1$ (type II)

it diverges

<u>Sol</u> $= \lim_{c\to 1^-} \int_0^c \frac{1}{1-x}\,dx$

<u>let $y = 1-x$, $dx = -dy$</u>

$= \lim_{c\to 1^-} - \int_{y=1}^{1-c} \frac{dy}{y} = \lim_{c\to 1^-} - \ln y \Big|_1^{1-c}$

$= \lim_{c\to 1^-} - \ln(1-c) = +\infty$ (diverges)

Eg6: $\int_0^\infty \frac{1}{x^p}\,dx$ converges? $(P>0)$

<u>Ans</u>: $\overline{\text{Type I}}$ at $[1,\infty)$

+ $\overline{\text{Type II}}$ at $(0,1]$

$$\int_0^\infty = \int_0^1 + \int_1^\infty$$

$$= \begin{cases} \text{finite} + \infty & 0<p<1 \\ +\infty + \infty & p=1 \\ +\infty + \text{finite} & p>1 \end{cases}$$

$=$ divergent (at least one of $\int_0^1$ or $\int_1^\infty$ diverges)

Eg 7. $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} dx$ converges?

<u>Sol</u>: Check $\int_0^1$ and $\int_1^3$ separately. (both type II)

From previous remark

both $\int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}}$ and $\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$

converge by the change of variable. $y=1-x$ or $y=x-1$

and $p=\frac{2}{3}$ (type II) $\Rightarrow \int_0^3$ converges