

Trigonometric integrals (8.3)

$$\int \cos^m x \sin^n x dx \quad \left(\int C^m S^n dx \right)$$

$$\text{if } m=2k+1 \quad \int C^{2k} S^n dx = \int (1-S^2)^k S^n dS$$

$$\text{if } n=2l+1 \quad \int C^m S^{2l} dx = \int C^m (1-C^2)^l dC$$

If $m=2k, n=2l$.

Method 1 (k, l small), $C_2 = \cos 2x$

$$\int C^{2k} S^{2l} dx = \int \left(\frac{1+C_2}{2} \right)^k \left(\frac{1-C_2}{2} \right)^l dx$$

$$= \int (A_0 + A_1 C_2 + A_2 C_2^2 + \dots + A_{k+l} C_2^{k+l}) dx$$

(reduce from $C^{2k} S^{2l}$ to C_2^{k+l})

Method 2 ($m=2k, n=2l$)

Case 1: $k \leq 1, l \leq 1 \rightarrow$ Method 1

Assume $k > 1$, we will

derive formula $\int C^{2k} S^{2l} dx \rightarrow \int C^{2k-2} S^{2l} dx$

$$= \int C^{2k-1} S^{2l} C dx = \int C^{2k-1} S^{2l} dS$$

$$= \frac{1}{2l+1} \int C^{2k-1} dS^{2l+1}$$

$$= \frac{1}{2l+1} \left(C^{2k-1} S^{2l+1} - \int S^{2l+1} dC^{2k-1} \right)$$

$$(*) = (2k-1) \int S^{2l+1} C^{2k-2} dC$$

= $-S^{2l+1} dx$

$$\therefore \int C^{2k} S^{2l} dx$$

$$= \frac{1}{2l+1} \left(C^{2k-1} S^{2l+1} + (2k+1) \left(\int C^{2k-2} S^{2l} dx - \int C^{2k} S^{2l} dx \right) \right)$$

$(S^2 = 1 - C^2)$

$$\Rightarrow \left(1 + \frac{2k-1}{2l+1} \right) \int C^{2k} S^{2l} dx$$

$$= \frac{1}{2l+1} C^{2k-1} S^{2l+1} + \frac{2k-1}{2l+1} \int C^{2k-2} S^{2l} dx$$

Reduce from $\int C^{2k} S^{2l} dx$ to $\int C^{2k-2} S^{2l} dx$

Recall. ($C = \cos x$, $S = \sin x$)
($C_2 = \cos 2x$, etc.)

$m, n, k, l > 0$

$$\int C^m S^{2l+1} dx = - \int C^m (1-C^2)^l dC$$

$$\int \frac{S^{2l+1}}{C^m} dx = - \int \frac{(1-C^2)^l}{C^m} dC$$

$$\int C^{2k+1} S^n dx = \int (1-S^2)^k S^n dS$$

$$\int \frac{C^{2k+1}}{S^n} dx = \int \frac{(1-S^2)^k}{S^n} dS$$

Rm $\int \frac{C^m}{S^{2l+1}} dx = \int \frac{C^m S dx}{S^{2l+2}}$

$$= - \int \frac{C^m}{(1-C^2)^{l+1}} dC \quad \left(\begin{array}{l} \text{Section 8.5} \\ \text{partial fractions} \end{array} \right)$$

Similarly for $\int \frac{S^n}{C^{2k+1}} dx$

If both m, n are even

$$\int C^m S^n dx = \frac{C^{m+1} S^{n-1}}{m+n} + \frac{n-1}{m+n} \int C^m S^{n-2} dx$$

$$\left(\begin{array}{c} \text{Table} \\ \text{in textbook} \end{array} \right) = \frac{C^{m-1} S^{n+1}}{m+n} + \frac{m-1}{m+n} \int C^{m-2} S^n dx$$

Important identities

$$(i) C_2 = 2C^2 - 1 = 1 - 2S^2$$

$$\Rightarrow 1 + C_2 = 2C^2, \quad 1 - C_2 = 2S^2$$

$$(ii) \int S_m S_n dx, \int S_m C_n dx, \int C_m C_n dx$$

$$\text{Use } 2S_m S_n = C_{m-n} - C_{m+n}$$

$$2S_m C_n = S_{m-n} + S_{m+n}$$

$$2C_m C_n = C_{m-n} + C_{m+n}$$

Remark

$$(i) S_2 = 2SC$$

$$(ii) \sqrt{1 + \sin x} = ?$$

$$\begin{aligned} & \left(\begin{aligned} \sin\left(x \pm \frac{\pi}{2}\right) &= \pm \cos x \\ \cos\left(x \pm \frac{\pi}{2}\right) &= \mp \sin x \end{aligned} \right. \\ &= \sqrt{1 - \cos\left(x - \frac{\pi}{2}\right)} \end{aligned}$$

$$\begin{aligned} &= \sqrt{2 \sin^2\left(\frac{x}{2} - \frac{\pi}{4}\right)} \\ &= \pm \sqrt{2} \sin\left(\frac{x}{2} - \frac{\pi}{4}\right) \end{aligned}$$

Similarly

$$\begin{aligned} \sqrt{1 - \sin x} &= \sqrt{1 + \cos\left(x - \frac{\pi}{2}\right)} \\ &= \pm \sqrt{2} \cos\left(\frac{x}{2} - \frac{\pi}{4}\right) \end{aligned}$$

$$\text{Ex 1 } \int S^2 C^4 dx = ?$$

$$(\text{Method I}) \quad m=4, n=2$$

$$= \frac{C^3 S^3}{4+2} + \frac{4-1}{4+2} \int C^2 S^2 dx$$

$$= \frac{C^3 S^3}{6} + \frac{1}{2} \cdot \frac{1}{4} \int (2CS)^2 dx$$

$$= \frac{C^3 S^3}{6} + \frac{1}{8} \int S_2^2 dx$$

$$= \frac{C^3 S^3}{6} + \frac{1}{8} \int \frac{1-C_4}{2} dx$$

$$= \frac{C^3 S^3}{6} + \frac{1}{16} \left(x - \frac{S_4}{4} \right) + \tilde{C}$$

$$= \frac{\cos^3 x \sin^3 x}{6} + \frac{1}{16} \left(x - \frac{\sin 4x}{4} \right) + \tilde{C}$$

Method II

$$C^4 = \left(\frac{1+C_2}{2}\right)^2 = \frac{1}{4} \left(1 + 2C_2 + \frac{1+C_4}{2}\right)$$

$$S^2 = \frac{1-C_2}{2}$$

$$C^4 S^2 = \frac{1}{16} (1-C_2) (3+4C_2+C_4)$$

$$= \frac{1}{16} \left(\begin{array}{l} 3 + 4C_2 + C_4 \\ - 3C_2 - 4C_2^2 - C_2C_4 \end{array} \right)$$

$$= \frac{1}{16} \left(3 + C_2 + C_4 - 2(1+C_4) - \frac{1}{2}(C_2+C_6) \right)$$

$$= \frac{1}{16} \left(1 + \frac{C_2}{2} - C_4 - \frac{C_6}{2} \right)$$

$$Ans = \frac{1}{16} \left(x + \frac{\sin 2x}{4} - \frac{\sin 4x}{4} - \frac{\sin 6x}{12} \right) + C$$

R_m : After reduction

$$\begin{matrix} (m, n) \\ = (2, 0) \end{matrix} \quad \int C^2 dx = \int \frac{1+C_2}{2} dx$$

$$(0, 2) \quad \int S^2 dx = \int \frac{1-C_2}{2} dx$$

$$(2, 2) \quad \int C^2 S^2 dx = \frac{1}{4} \int S_2^2 dx$$

$$= \frac{1}{4} \int \frac{1-C_4}{2} dx$$

$$\int t^m e^n dx : t = \tan x, e = \sec x$$

$$dt = e^2 dx \quad de = t e dx$$

$$\begin{aligned} \text{(i)} \quad \int t^{2k+1} e^n dx &= \int t^{2k} e^{n-1} (t e dx) \\ &= \int (e^2 - 1)^k e^{n-1} de \end{aligned}$$

$$R_m: \int t^{2k+1} e^n dx = \int \frac{S^{2k+1}}{C^{2k+1+n}} dx = - \int \frac{(1-c^2)^k}{C^{2k+1+n}} dc$$

$$\begin{aligned} \text{(ii)} \quad \int t^m e^{2l} dx &= \int t^m e^{2l-2} (e^2 dx) \\ &= \int t^m (t^2 + 1)^{l-1} dt \quad (m \text{ can be even or odd}) \end{aligned}$$

$$\text{However } \int t^m e^{2l} dx = \int \frac{S^m}{C^{m+2l}} dx$$

not easy if $m = \text{even}$

$$(iii) \int t^{2k+1} dx = \int \frac{t^{2k+1}}{e} dx = \int \frac{(e^2 - 1)^k}{e} de$$

$$\int \frac{t^{2k+1}}{c^{2k+1}} dx = - \int \frac{(1-c^2)^k}{c^{2k+1}} dc$$

$$(iv) \int t^{2k} dx = \int \frac{t^{2k}}{e^2} dx = \int \frac{t^{2k} dt}{1+t^2}$$

$$= \int \left(\text{Polynomial of } t^2 + \frac{\hat{C}}{1+t^2} \right) dt$$

$$= P(t^2) + \hat{C} \tan^{-1}(\tan x) + \tilde{C}$$

$$\alpha = \int t^{2k-2} (e^2 - 1) dx = \int t^{2k-2} (e^2 dx) - \int t^{2k-2} dx$$

$$= \int t^{2k-2} dt - \int t^{2k-2} dx \left(\text{reduce from } \int t^{2k} dx \text{ to } \int t^{2k-2} dx \right)$$

$$\int t^{2k} e^{2l+1} dx \rightarrow \text{integration by parts}$$

In general

$$\int t^m e^n dx \begin{cases} \rightarrow \int t^{m-2} e^n dx & (1) \\ \rightarrow \int t^m e^{n-2} dx & (2) \end{cases}$$

For (1): ($t^2 dx = de$)

$$= \int t^{m-1} e^n de = \frac{1}{n} \int t^{m-1} de^n$$

$$= \frac{1}{n} \left(t^{m-1} e^n - \int e^n (m-1) \underbrace{t^{m-2}}_{\frac{d}{dt} t^{m-1}} dx \right)$$

$$= \frac{1}{n} \left(t^{m-1} e^n - (m-1) \int t^m e^n dx - (m-1) \int t^{m-2} e^n dx \right)$$

$$\Rightarrow \int t^m e^n dx = \frac{1}{n+m-1} t^{m-1} e^n - \frac{m-1}{n+m-1} \int t^{m-2} e^n dx$$

Similarly for (2). ($e^2 dx = dt$)

$$= \frac{1}{n+m-1} t^{m-1} e^{n-2} + \frac{(n-2)}{n+m-1} \int t^m e^2 dx$$

$$(z_k, z_{l+1}) \begin{matrix} \nearrow (z_k, z_{l-1}) \\ \searrow (z_{k-2}, z_{l+1}) \end{matrix} \rightarrow \dots (2, 1) \rightarrow (0, 1)$$

$$\int t^2 e' dx = \int \frac{s^2}{c^3} dx = \int \frac{s^2 c dx}{c^4} = \int \frac{s^2 ds}{(1-s^2)^2} \quad (8.5)$$

$$\downarrow$$

$$\int e dx = \int \frac{1}{c} dx = \int \frac{c dx}{c^2} = \int \frac{ds}{1-s^2}$$

$$= \frac{1}{2} \int \left(\frac{1}{s+1} - \frac{1}{s-1} \right) dx$$

$$= \frac{1}{2} \ln \left| \frac{1+\sin x}{\sin x-1} \right| + \tilde{C}$$

$$= \ln \sqrt{\frac{1+\sin x}{1-\sin x}} + \tilde{C}$$

$$\text{or } \int e dx = \int \frac{e(t+e)}{t+e} = \int \frac{d(t+e)}{t+e}$$

$$= \ln |\tan x + \sec x| + \tilde{C}$$

$$t = \tan x, e = \sec x$$

$$\text{Eq 1 } \int t^3 e^2 dx = \int t^3 dt$$

$$\alpha = \int t^2 e de = \int (e^2 - 1) e de$$

$$\alpha = \int \frac{S^3}{C^5} dx = - \int \frac{(4 - C^2) dC}{C^5}$$

$$\text{Eq 2 } \int t^2 e dx$$

$$= \int t de = e \cdot t - \int e dt$$

$$= et - \int e^3 dx = et - \int (t^2 + 1) e dx$$

$$= et - \int t^2 e dx - \int e dx$$

$$\Rightarrow \int t^2 e = \frac{et}{2} - \frac{1}{2} \int \frac{1}{\cos x} dx = \dots$$

$$\text{Ex 3 } \int e^6 dx$$

$$= \int e^4 dt = \int (t^2 + 1)^2 dt$$

$$= \int (t^4 + 2t^2 + 1) dt$$

$$\text{Ex 4 } \int t^4 dx$$

$$= \int t^2 (e^2 - 1) dx = \int t^2 dt - \int t^2 dx$$

$$= \int t^2 dt - \int (e^2 - 1) dx$$

$$= \int t^2 dt - \int dt - \int dx = \dots$$

$$\text{Ex 5 } \int t^5 dt \rightarrow \int t^3 dt \rightarrow \int t dt$$

$$or = \int \frac{s^5}{c^5} dx = - \int \frac{s^4 dc}{c^5} = - \int \frac{(1-c^2)^2}{c^5} dc$$