Rate of Growth

Def

$f(x)$ grows $\begin{cases} \text{faster than} \\ \text{at the same rate as} \\ \text{slower than} \end{cases}$ $g(x)$

as $x \to \infty$

if $\lim\limits_{x \to \infty} \dfrac{|f(x)|}{|g(x)|} = \begin{cases} \infty \\ L, & 0 < L < \infty \\ 0 \end{cases}$

Eg 1 $-5x^5+7x^4-2x^2-1$

grows at the same rate

as x^5, as $x \to \infty$

Eg 2: $f(x) = e^{0.01x}$

$g(x) = x^7$, $h(x) = (\ln x)^{1000}$

$f(x)$ grows faster than $g(x)$

$g(x)$ grows faster than $h(x)$

as $x \to \infty$

Since $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$, $\lim_{x\to\infty} \left(\frac{h(x)}{g(x)}\right)^{\frac{1}{1000}} = 0$,

Def. $f(x) = o(g(x))$ as $x \to \infty$, 0^+, 0, ...

if $\lim\limits_{\substack{x \to \infty \\ 0, 0^+}} \frac{f(x)}{g(x)} = 0$

(i.e. f "$<$" g)

Def $f(x) = O(g(x))$ as $x \to \infty$, 0, 0^+ ..

(if $\left|\frac{f(x)}{g(x)}\right| \leq M$ as $x \to \infty, 0, 0^+$)

if there exists $M > 0$, $N > 0$, ($\delta > 0$)

such that $\left|\frac{f(x)}{g(x)}\right| \leq M$ for all $x > N$, $x \in (-\delta, \delta)$, $x \in (0, \delta)$

(i.e. f "$\leq$" g)

Remark

(i) $\lim \frac{f}{g} = 0 \iff f = o(g)$

(ii) $\lim \left|\frac{f}{g}\right| = \underset{\neq 0}{L} \;\substack{\Longrightarrow \\ \not\Longleftarrow}\; \begin{cases} f = O(g) \\ g = O(f) \end{cases}$

(iii) $\lim \left|\frac{f}{g}\right| = \begin{cases} 0 \\ L \end{cases} \Rightarrow f = O(g)$

$(f = o(g) \Rightarrow f = O(g))$

Eg. $f(x) = x, \quad g(x) = x(2+\sin x)$

$\Rightarrow \left|\frac{f(x)}{g(x)}\right| = 2+\sin x \;\substack{\leq 3 \\ \geq 1} \Rightarrow \begin{cases} f = O(g) \\ g = O(f) \end{cases}$

But $\lim\limits_{x\to\infty} \left|\frac{f(x)}{g(x)}\right|$ does not exist as $x \to \infty$

Eg 3

$$-5x^5+7x^4-2x-1=O(x^5)$$

as $x \to \infty$

Eg 4 $$(\ln x)^{1000} = o(x^7)$$

$$x^7 = o(e^{0.001x}) \quad \text{as } x \to \infty$$

Eg 5 $$(\ln x)^{1000} = o(x^{-2})$$

$$x^{-2} = o\left(e^{\frac{1}{x^2}}\right) \quad \text{as } x \to 0^+$$

Eg 6 $x^p = o(x^q)$

(i) as $x \to \infty$ if $p < q$

(ii) as $x \to 0$ if $p > q$

Eg 7. $x + \sin x = O(x)$ as $x \to \infty$

Eg 8: $e^x + x^3 = O(e^x)$

as $x \to \infty$

Techniques of integration

8.2 Integration by parts.

$$\int f(x)\,g'(x)\,dx = \int f(x)\,dg(x) = ?$$

Ans: $\frac{d}{dx}\int fg'\,dx = fg' = (fg)' - f'g$

$$= \frac{d}{dx}\left(fg - \int f'g\,dx\right)$$

$$\Rightarrow \int fg'\,dx = fg - \int f'g\,dx$$

or $\int f\,dg = fg - \int g\,df$

Useful if $\int g\,df = \int gf'\,dx$ is easy to compute.

Eg1 $\int \overset{\overset{g'}{f}}{x} \overset{\overset{f}{g'}}{\cos x}\, dx$

$$= \begin{cases} \int x \, d\sin x = x\sin x - \int \sin x \, dx \\ \int \cos x \, d\frac{x^2}{2} = \underbrace{\frac{x^2}{2}\cos x - \underbrace{\int \frac{x^2}{2} \, d\cos x}_{\text{worse}}}_{\text{NG}} \end{cases}$$

$$= x\sin x + \cos x + C$$

Check: $\frac{d}{dx}(x\sin x + \cos x)$

$$= \sin x + x\cos x - \sin x = \cos x$$

(correct)

Eg 2 $\int \underbrace{\ln x}_{f} \underbrace{dx}_{dg}$, $(x>0)$

$$= x\ln x - \int x \, d\ln x$$

$$= x\ln x - \int x \frac{1}{x} dx$$

$$= x\ln x - x + C$$

check: $\frac{d}{dx}(x\ln x - x)$

$$= x \cdot \frac{1}{x} + 1 \cdot \ln x - 1$$

$= \ln x$ (correct)

Eg3. $\int x^2 e^x dx$

$$= \int x^2 de^x$$

$$= x^2e^x - \int e^x dx^2$$

$$= x^2e^x - 2\int xe^x dx$$

$$= x^2e^x - 2\int x de^x$$

$$= x^2e^x - 2\left(xe^x - \int e^x dx\right)$$

$$= x^2e - 2xe^x + 2e^x + C$$

Eg4 $\int x^2 \ln x \, dx$, $x>0$

$$= \int \ln x \, d\frac{x^3}{3}$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \, d\ln x$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \frac{1}{x} \, dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

Check:

$$\frac{d}{dx} = x^2 \ln x + \frac{x^3}{3}\cdot\frac{1}{x} - \frac{x^2}{3}$$

$$= x^2 \ln x \text{ (correct)}$$

Eg 5 $\int \cos^n x \, dx$, $n \geq 1$

Case 1. $n = 2k+1$.

$= \int \cos^{2k} x \cos x \, dx$

$= \int (1-\sin^2 x)^k \, d\sin x$

$= \int (1-s^2)^k \, ds$

If $n = $ even $\quad c = \cos x$, $s = \sin x$

$= \int c^{n-1} c \, dx = \int c^{n-1} \, ds$

$= c^{n-1} s - \int s \, d(c^{n-1})$

$= c^{n-1} s - (n-1) \int s \, c^{n-2} \overbrace{(-s) \, dx}^{dc}$

$$= C^{n-1}S + (n-1)\int C^{n-2}(1-C^2)dx$$

$$\Rightarrow \int C^n dx = C^{n-1}S + (n-1)\int C^{n-2}dx - (n-1)\int C^n dx$$

$$\Rightarrow n\int C^n dx = C^{n-1}S + (n-1)\int C^{n-2}dx$$

$$\Rightarrow \int \cos^n x\,dx = \frac{\cos^{n-1}x\,\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

$$\int C^n dx \to \int C^{n-2}dx \xrightarrow{\cdots} \int C^2 dx \to \int 1\,dx$$

Similarly. (n = even)

$$\int S^n dx \to \int S^{n-2}dx \xrightarrow{\cdots} \int S^2 dx \to \int 1\,dx$$

Definite integrals

$$fg' = (fg)' - gf'$$

$$\int_a^b dx$$

$$\Rightarrow \int_a^b fg'\,dx = fg\Big|_a^b - \int_a^b gf'\,dx$$

or $$\int_{x=a}^b f\,dg = fg\Big|_a^b - \int_{x=a}^b g\,df$$

Eg6: $$\int_0^4 xe^x\,dx = \int_{x=0}^4 x\,de^x$$

$$= xe^x\Big|_0^4 - \int_{x=0}^4 e^x\,dx$$

$$= xe^x\Big|_0^4 - e^x\Big|_0^4 = 3e^4 + 1$$

Trigonometric integrals

$$\int \cos^m x \sin^n x\, dx \quad \left(\int c^m s^n dx\right)$$

If $m=2k+1$ $\int c^{2k} s^n c\, dx = \int (1-s^2)^k s^n ds$

if $n=2l+1$ $\int c^m s^{2l} s\, dx = -\int c^m (1-c^2)^l dc$

If $m=2k$, $n=2l$

Method 1 (k, l small), $C_2 \equiv \cos 2x$

$$\int c^{2k} s^{2l} dx = \int \left(\frac{1+C_2}{2}\right)^k \left(\frac{1-C_2}{2}\right)^l dx$$

$$= \int (a_0 + a_1 C_2 + a_2 C_2^2 + \cdots + a_{k+l} C_2^{k+l})\, dx$$

(reduce from $c^{2k} s^{2l}$ to C_2^{k+l}

Method 2 ($m=2k$, $n=2\ell$)

Case 1: $k\le 1$, $\ell \le 1 \to$ Method 1

Assume $k>1$, we will

derive formula $\int C^{2k} S^{2\ell} dx \to \int C^{2k-2} S^{2\ell} dx$

$$=\int C^{2k-1} S^{2\ell} C\,dx = \int C^{2k-1} S^{2\ell}\, dS$$

$$=\frac{1}{2\ell+1}\int C^{2k-1}\, dS^{2\ell+1}$$

$$=\frac{1}{2\ell+1}\left(C^{2k-1} S^{2\ell+1} - \int \underbrace{S^{2\ell+1}\, dC^{2k-1}}_{(*)} \right)$$

$$(*) = (2k-1)\int S^{2\ell+1} C^{2k-2} \underset{\underset{-S\,dx}{\parallel}}{dC}$$

$$\therefore \int C^{2k} S^{2l}\,dx$$

$$= \frac{1}{2l+1}\left(C^{2k-1} S^{2l+1} + (2k+1)\left(\int C^{2k-2} S^{2l} dx - \int C^{2k} S^{2l} dx \right)\right)$$

$$(S^2 = 1 - C^2)$$

$$\Rightarrow \left(1 + \frac{2k-1}{2l+1}\right) \int C^{2k} S^{2l}\,dx$$

$$= \frac{1}{2l+1} C^{2k-1} S^{2l+1} + \frac{2k-1}{2l+1} \int C^{2k-2} S^{2l}\,dx$$

Reduce from $\int C^{2k} S^{2l}\,dx$ to $\int C^{2k-2} S^{2l}\,dx$