Hyperbolic functions

(SKIP inverse hyperbolic fcns)

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \begin{pmatrix}\text{hyperbolic} \\ \text{sine}\end{pmatrix}$$

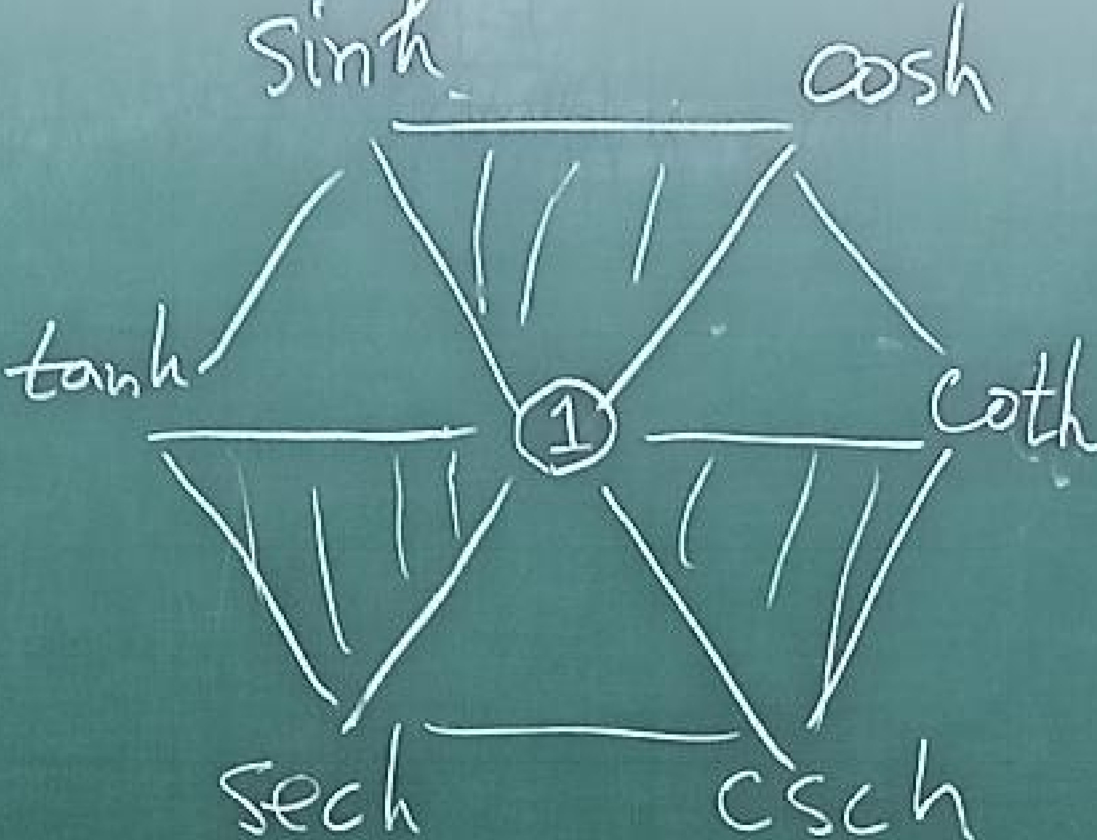
$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \begin{pmatrix}\text{hyp.} \\ \text{cosine}\end{pmatrix}$$

$$\tanh x = \sinh x / \cosh x$$

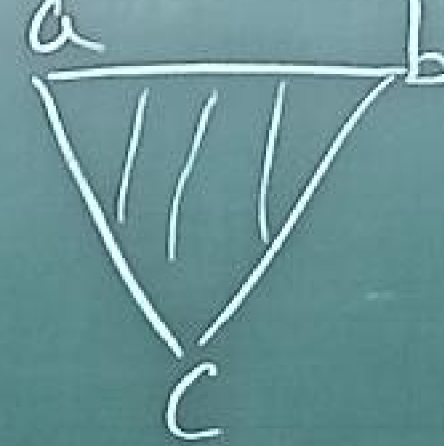
$$\coth x = \cosh x / \sinh x \quad (x \neq 0)$$

$$\operatorname{sech} x = 1/\cosh x$$

$$\operatorname{csch} x = 1/\sinh x \quad (x \neq 0)$$



(i)



$$b^2 - a^2 = 1$$

(ii) Diagnal entries are reciprocal to each other

(iii) Any corner entry is the product of two neighboring corner entries

Other algebraic identities

$$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$$

$$= \frac{(e^{x}+e^{-x})(e^{x}-e^{-x})}{2} = 2\sinh x \cosh x$$

$$\cosh 2x = \frac{e^{2x} + e^{-2x}}{2}$$

$$= \frac{1}{2}\left((e^{x}+e^{-x})^{2}-2\right) = 2\cosh^{2}x - 1$$

$$= \frac{1}{2}\left((e^{x}-e^{-x})^{2}+2\right) = 2\sinh^{2}x + 1$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

Remark

$$u'' + k^2 u = 0 \Rightarrow u(x) = \begin{matrix} \sin kx \\ \cos kx \end{matrix}$$

$$\begin{cases} u'' + k^2 u = 0 \\ u(0) = 0 \ (1) \\ u'(0) = 1 \ (0) \end{cases} \Rightarrow u(x) = \frac{1}{k}\sin kx$$

$$(\cos kx)$$

$$u'' - k^2 u = 0 \Rightarrow u(x) = e^{\pm kx}$$

$$(\text{or } \sinh x, \cosh x)$$

$$\begin{cases} u'' - k^2 u = 0 \\ u(0) = 0 \ (1) \\ u'(0) = 1 \ (0) \end{cases} \Rightarrow u(x) = \frac{1}{k}\sinh kx$$

$$(\cosh kx)$$

Derivative of hyperbolic fcns

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\coth x = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = \frac{d}{dx}(\cosh x)^{-1} = \frac{-\sinh x}{\cosh^2 x} = -\tanh x \operatorname{sech} x$$

$$\frac{d}{dx}\operatorname{csch} x = \frac{d}{dx}(\sinh x)^{-1} = \frac{-\cosh x}{\sinh^2 x} = -\coth x \operatorname{csch} x$$

(See also table 7.6)

Eg1 $\frac{d}{dx}\tanh\sqrt{1+x^2}$

$=\left(\operatorname{sech}^2\sqrt{1+x^2}\right)\cdot\frac{x}{\sqrt{1+x^2}}$

Eg2 $\int_0^1 \tanh x\,dx$

$=\int_0^1 \frac{\sinh x}{\cosh x}\,dx$

$=\int_0^1 \frac{d\cosh x}{\cosh x}$

$=\ln|\cosh x|\Big|_0^1$

$=\ln(\cosh 1)-\ln 1=\ln\left(\frac{e+e^{-1}}{2}\right)$

Eg3 $\int \sinh^2 x\,dx$

$\sinh^2 x=\frac{(e^{x}-e^{-x})^2}{4}$

$=\frac{e^{2x}-2+e^{-2x}}{4}$

$=\frac{1}{2}\left(\left(\frac{e^{2x}+e^{-2x}}{2}\right)-1\right)$

$=\frac{1}{2}(\cosh 2x-1)$.

$\therefore \text{Ans}=\frac{1}{2}\int(\cosh 2x-1)\,dx$

$=\frac{1}{2}\left(\frac{1}{2}\sinh 2x-x\right)+C$

$=\frac{1}{4}\sinh 2x-\frac{x}{2}+C$

Eg 4 $\int_0^{\ln 2} 4e^x \sinh x \, dx$

$$= \int_0^{\ln 2} 4 \cdot e^x \cdot \frac{e^x - e^{-x}}{2} \, dx$$

$$= 2\int_0^{\ln 2} e^{2x} - 1 \, dx$$

$$= 2\left(\frac{e^{2x}}{2} - x\right)\Big|_0^{\ln 2}$$

$$= \left(e^{2\ln 2} - 1\right) - 2(\ln 2 - 0)$$

$$= 3 - 2\ln 2$$