

Separable differential equations

First order differential equations

Solve for $y(x)$ from

$$\frac{dy(x)}{dx} = F(x, y(x))$$

Eg: $\begin{cases} y'(x) = \sin(x+y) \\ y(0) = y_0 \end{cases}$

Special cases

(I): Separable: $F(x, y) = g(x)H(y)$

(II): linear: $F(x, y) = -P(x)y + Q(x)$
(linear in y)

Ex 1 (Separable) Solve $y(x)$ from

$$\frac{dy}{dx} = (1+y)e^x \quad (y > -1)$$

Ans: $\frac{\frac{dy}{dx}}{1+y} = e^x$

$$\frac{dy}{1+y} = e^x dx$$

$$\int dx \Rightarrow \int \frac{\frac{dy}{dx}}{1+y} dx = \int e^x dx$$

$$\text{i.e. } \int \frac{dy}{1+y} = \int e^x dx$$

$$\ln(1+y)^{y>-1} = \ln|1+y| = e^x + C$$

$$\Rightarrow 1+y = e^{(e^x+C)} = C_1 e^{e^x} \quad (C_1 = e^C)$$

$$\therefore y = C_1 e^{e^x} - 1 \quad (C_1 \text{ to be determined from additional BC})$$

Eq 2: Solve $y \cdot (x+1) \frac{dy}{dx} = x(y^2+1)$

$$\left(\frac{y}{y^2+1} dy = \frac{x}{x+1} dx \right)$$

Sol $\int \frac{y}{1+y^2} dy = \int \frac{x}{x+1} dx$

$$\Rightarrow \frac{1}{2} \int \frac{dy^2}{1+y^2} = \int \left(1 - \frac{1}{x+1} \right) dx$$

$$\Rightarrow \frac{1}{2} \int \frac{d(1+y^2)}{1+y^2} = x - \int \frac{dx}{x+1}$$

$$\Rightarrow \frac{1}{2} \ln(1+y^2) = x - \ln|1+x| + C$$

$$\Rightarrow \ln(1+y^2) = 2x - \ln(1+x)^2 + 2C$$

$$y = \pm \sqrt{\frac{e^{2(x+C)} - 1}{(1+x)^2}} \quad \left(\begin{array}{l} \text{2 sets of} \\ \text{solutions} \end{array} \right)$$

First order linear differential equations. (textbook: $P, Q \rightarrow \mathbb{R}, \mathbb{Q}$)

$$\frac{dy}{dx} = \underbrace{-p(x)y + q(x)}_{\text{linear in } y}$$

(Solve for $y(x)$)

Eg1: $y' = 3y, y(0) = 2$

Sol: $e^{-3x} \cdot (y' - 3y = 0)$

$$\Rightarrow (e^{-3x} y)' = 0, e^{-3x} y(x) = C$$

$$y(0) = 2 \Rightarrow C = 2, \therefore y(x) = 2e^{3x}$$

In general

$$\frac{dy}{dx} + p(x)y = q(x), \quad y(a) = y_0$$

Multiply by $e^{P(x)}$, where $P'(x) = p(x)$

$$(e^{P(x)} y)' = e^{P(x)} q(x)$$

$$\therefore e^{P(x)} y(x) = \int e^{P(x)} q(x) dx = \int_a^x e^{P(t)} q(t) dt$$

$y(a) = y_0 \Rightarrow$ find the undetermined constant in $\int e^{P(x)} q(x) dx$

or $\int_a^x (e^{P(t)} y(t))' dt = \int_a^x e^{P(t)} q(t) dt$

$$\therefore e^{P(x)} y(x) - e^{P(a)} y(a) = \int_a^x e^{P(t)} q(t) dt$$

$$\therefore y(x) = e^{-(P(x)-P(a))} y_0 + e^{-P(x)} \int_a^x e^{P(t)} q(t) dt$$

Ex2 Solve $e^x y' + 2e^{2x} y = 1$

Sol: $\frac{dy}{dx} + 2e^x y = e^{-x}$

Multiply by e^{2e^x}

$$\Rightarrow (e^{2e^x} y)' = e^{2e^x - x}$$

$$\therefore e^{2e^x} y(x) = \int e^{2e^x - x} dx$$

$$\begin{aligned} y(x) &= e^{-2e^x} \int e^{2e^x - x} dx \\ &= e^{-2e^x} \int^x e^{2e^t - t} dt \end{aligned}$$

Eg 3. Solve $\begin{cases} y' + xy = x \\ y(0) = 2 \end{cases}$

Sol Multiply by $e^{\frac{x^2}{2}}$

$$\left(e^{\frac{x^2}{2}} y \right)' = x e^{\frac{x^2}{2}}$$

$$\int_0^x dt$$

$$\int_0^x \left(e^{\frac{t^2}{2}} y(t) \right)' dt = \int_0^x t e^{\frac{t^2}{2}} dt$$

$$e^{\frac{x^2}{2}} y(x) - y(0) = \int_0^x e^{\frac{t^2}{2}} d\left(\frac{t^2}{2}\right)$$

$$e^{\frac{x^2}{2}} y(x) = y(0) + e^{\frac{x^2}{2}} - e^0$$

$$y(x) = e^{\frac{x^2}{2}} + 1$$

$$\text{Ex 4 } \begin{cases} (1+x)y' + y = \sqrt{x} \\ y(0) = 1 \end{cases} \quad (x \geq 0)$$

Sol $y' + \frac{1}{1+x}y = \frac{\sqrt{x}}{1+x}$

$$p(x) = \frac{1}{1+x}, \quad P(x) = \ln(1+x)$$

$$e^{P(x)} = (1+x)$$

$$((1+x)y)' = \sqrt{x}$$

$$\int_0^x [(1+t)y_t] dt = \int_0^x \sqrt{t} dt$$

$$(1+x)y(x) - (1+0)y(0) = \frac{2}{3}x^{\frac{3}{2}}$$

$$\therefore y(x) = \frac{\frac{2}{3}x^{\frac{3}{2}} + 1}{1+x}$$

$$\text{Ex 5} \quad \begin{cases} x \frac{dy}{dx} = x^2 + 3y, & x > 0 \\ y(1) = 1 \end{cases}$$

Sol $x y' - 3y = x^2$

$$y' - \frac{3}{x}y = x$$

$$\begin{aligned} (x^{-3}y)' &= -3x^{-4}y + x^{-3}y' \\ &= x^{-4}(x y' - 3y) \end{aligned}$$

$$x y' - 3y = x^4 (x^{-3}y)'$$

$$\Rightarrow (x^{-3}y)' = x^{-2}$$

$$\int_1^x dt \Rightarrow x^{-3}y(x) - 1 \cdot y(1) = \int_1^x t^{-2} dt$$

$$\Rightarrow y(x) = -x^2 + 2x^3$$

In general, the special case,
(k is an integer)

$$x y' + k y = g(x)$$

$$x^{(1-k)} \left(x^k y \right)'$$

$$\Rightarrow (x^k y)' = x^{k-1} g(x)$$

i.e. In this case

$$p(x) = \frac{k}{x} \Rightarrow P(x) = k \ln|x|$$

Integration factor

$$= e^{P(x)} = e^{k \ln|x|} = |x|^k = \pm x^k$$

$$\text{i.e. } x^k \left(y' + \frac{k}{x} y \right) = \frac{g(x)}{x}$$

Eg 6. $y' + 2y = 3$, $y(0) = 1$

Sol. $e^{2x}(y' + 2y) = 3e^{2x}$

$$\Rightarrow (e^{2x}y)' = 3e^{2x}$$

$$\int_0^x dt \Rightarrow e^{2x}y(x) - e^0y(0) = 3 \int_0^x e^{2t} dt$$

$$\begin{aligned} \therefore y(x) &= e^{-2x} \left(1 + 3 \cdot \frac{e^{2t}}{2} \Big|_0^x \right) \\ &= \frac{1}{2} e^{-2x} + \frac{3}{2} \end{aligned}$$