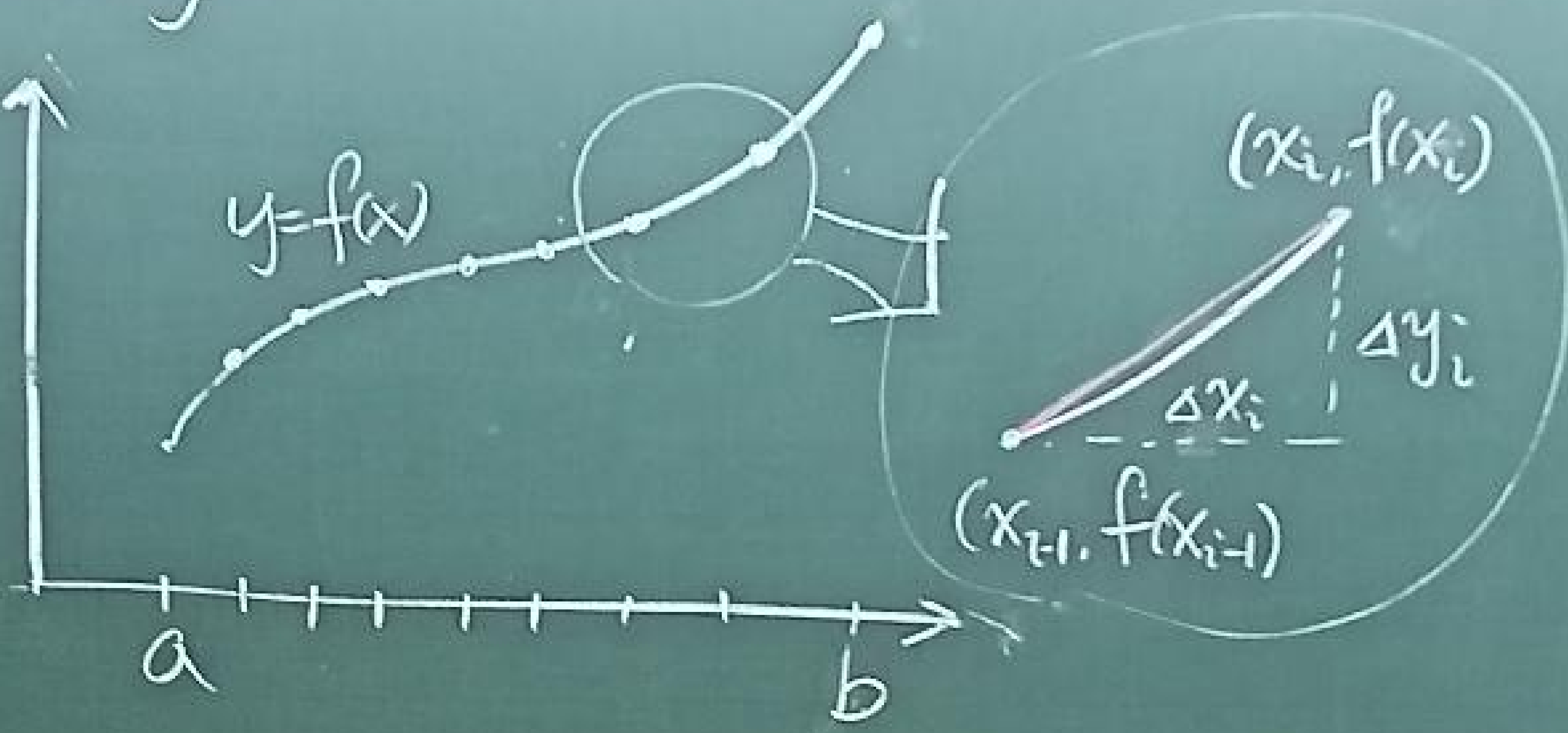
Arclength



$$L_i \cong \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$$

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i \qquad \begin{aligned} \Delta x_i &= x_i - x_{i-1} \\ \Delta y_i &= f(x_i) - f(x_{i-1}) \end{aligned}$$

$$= \int_{x=a}^{b} \sqrt{1 + (f'(x))^2}\,dx$$

Rm If the curve is

$$\{c \le y \le d,\ x = g(y)\}$$

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta x_i}{\Delta y_i}\right)^2}\, \Delta y_i$$

$$= \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

Rm If the curve is

$$\left\{\alpha \le t \le \beta,\ \begin{matrix} x = X(t) \\ y = Y(t) \end{matrix}\right\}$$

$$L_i \simeq \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\, \Delta t_i$$

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\, \Delta t_i = \int_\alpha^\beta \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)}\, dt$$

Eg1 Perimeter of $\{x^2+y^2=a^2\}$

Method 1 $y=\pm\sqrt{a^2-x^2}$

$$L=2\int_{-a}^{a}\sqrt{1+\left(\frac{d}{dx}\sqrt{a^2-x^2}\right)^2}\,dx$$

$$=2\int_{-a}^{a}\sqrt{1+\frac{x^2}{a^2-x^2}}\,dx$$

$$=2a\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}=2a\int_{x=-a}^{a}\frac{d\frac{x}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

$$=2\pi a$$

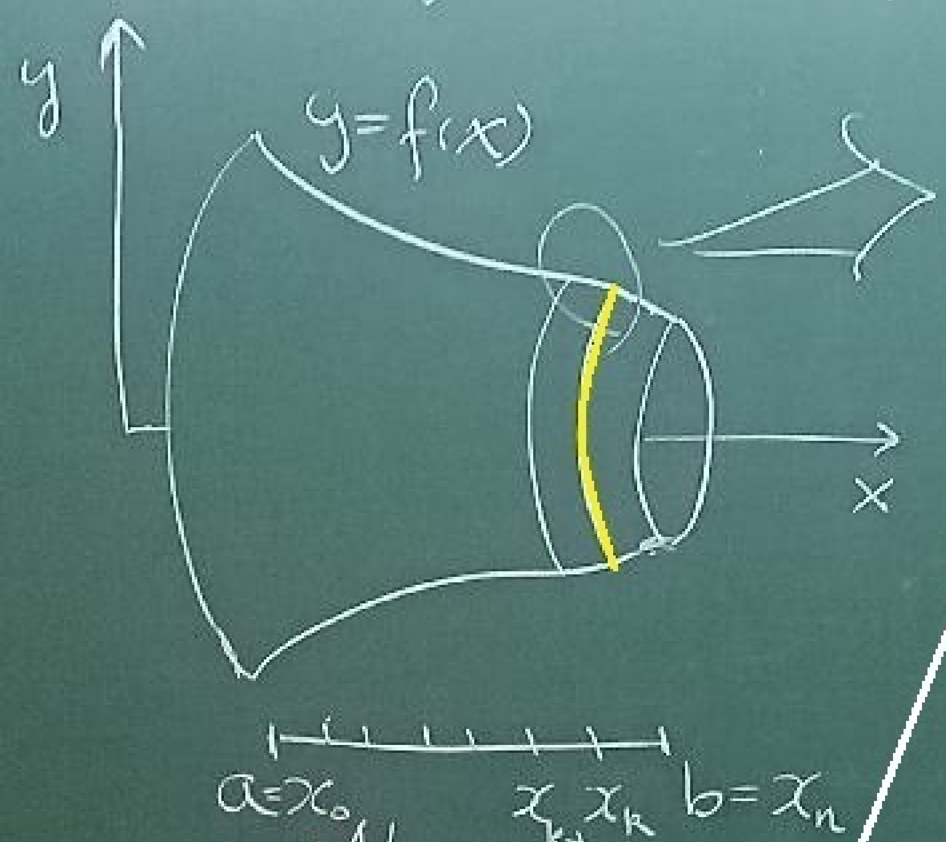
Method 2: $L=2\int_{y=-a}^{a}\sqrt{1+\frac{y^2}{a^2-y^2}}\,dy$

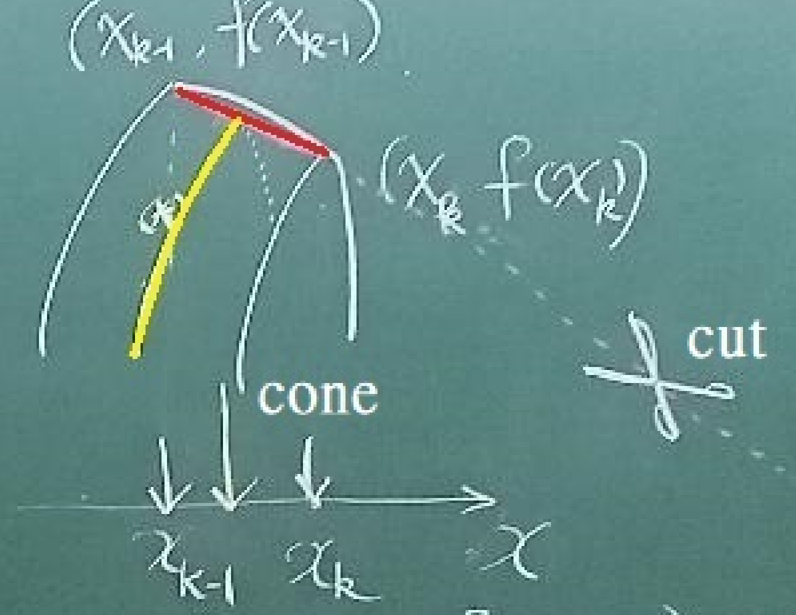
Method 3 $x=a\cos t,\ y=a\sin t,\ 0\le t\le 2\pi$

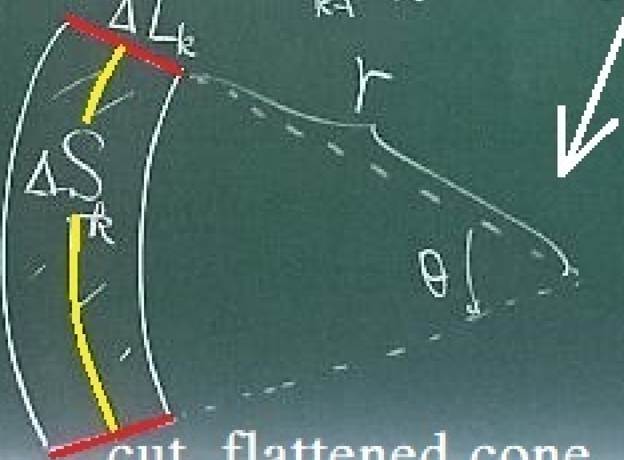
$$x'=-a\sin t,\ y'=a\cos t$$

$$L=\int_0^{2\pi}\sqrt{(a\sin t)^2+(a\cos t)^2}\,dt=2\pi a$$

Area of Surface of Revolution







$$\Delta S_k = \left(\pi(r+\Delta L_k)^2 - \pi r^2\right)\cdot\frac{\theta}{2\pi}$$

$$= \pi\left(2r\Delta L_k + \Delta L_k^2\right)\cdot\frac{\theta}{2\pi}$$

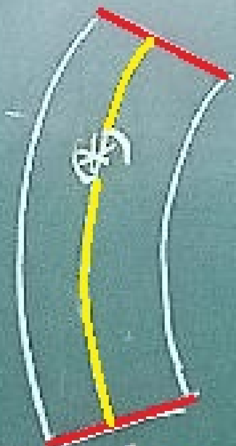
$$= \underline{\Delta L_k}\,\underline{\left(2\pi\left(r + \frac{\Delta L_k}{2}\right)\right)\frac{\theta}{2\pi}}$$

$$= \Delta L_k \cdot \tilde{L}_k$$

$$\Delta L_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

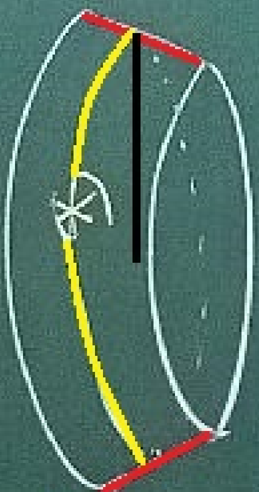
$$\tilde{L}_k = \text{arclength of } (*)$$

$$= 2\pi \underbrace{f\left(\frac{x_{k-1}+x_k}{2}\right)}_{\equiv c_k}$$



flattened cone

$$\|$$



cone

$$S = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta S_k$$

$$= \lim_{\|P\|\to 0} \sum_{k=1}^{n} 2\pi f(c_k)\, \Delta L_k$$

$$= \lim_{\|P\|\to 0} \sum_{k=1}^{n} 2\pi f(c_k) \sqrt{1+\left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\, \Delta x_k$$

$$= \int_a^b 2\pi \underbrace{f(x)}_{\text{半徑}} \underbrace{\sqrt{1+f'(x)^2}\, dx}_{\text{弧長元}}$$

<u>Remark</u> For a surface obtained by rotating $\left\{ \begin{matrix} c \le y \le d \\ x = g(y) \end{matrix} \right\}$ around y-axis, the surface

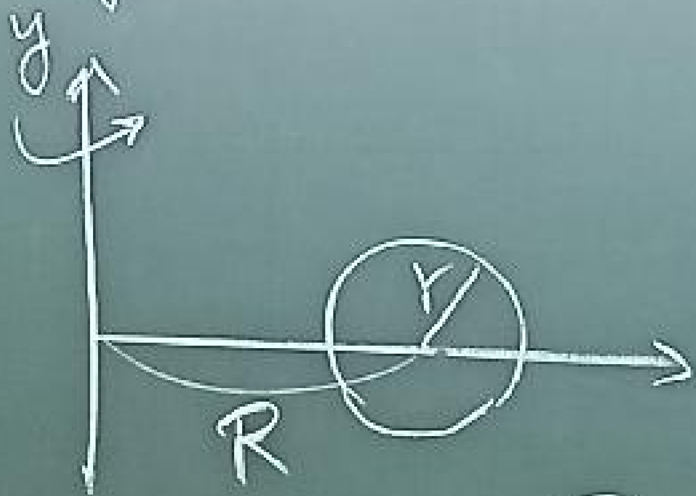
$$\text{area} = \int_c^d 2\pi g(y) \sqrt{1 + g'(y)^2}\, dy$$

<u>Rm</u> $\left\{ \alpha \le t \le \beta,\ \begin{matrix} x = X(t) \\ y = Y(t) \end{matrix} \right\}$

$$S_{\rightarrow} = \int_\alpha^\beta 2\pi Y(t) \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)}\, dt$$

$$S_{\uparrow} = \int_\alpha^\beta 2\pi X(t) \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)}\, dt$$

Eg1 Surface area of a donut



generated by rotating $\{(x-R)^2+y^2=r^2\}$ around y-axis.

半徑　弧長元

$$S^{上+下}=2\int_{x=R-r}^{R+r} 2\pi x \, d\ell$$

$$\left(y=\pm\sqrt{r^2-(x-R)^2}\right) \qquad d\ell=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\frac{dy}{dx}=\pm\left(-\frac{(x-R)}{\sqrt{r^2-(x-R)^2}}\right)$$

$$\therefore d\ell=\sqrt{1+\frac{(x-R)^2}{r^2-(x-R)^2}}\,dx=\frac{r}{\sqrt{r^2-(x-R)^2}}\,dx$$

$$\therefore S^{\text{上}+\text{下}} = 2\int_{R-r}^{R+r} 2\pi x \frac{r}{\sqrt{r^2-(x-R)^2}}\,dx$$

Let $\mathbb{X} = x - R \quad dx = d\mathbb{X}$

$$= 2\int_{\mathbb{X}=-r}^{r} 2\pi(\mathbb{X}+R)\frac{r}{\sqrt{r^2-\mathbb{X}^2}}\,d\mathbb{X}$$

$$= 4\pi r\left(\int_{-r}^{r}\frac{\mathbb{X}}{\sqrt{r^2-\mathbb{X}^2}}\,d\mathbb{X} + \int_{-r}^{r}\frac{R}{\sqrt{r^2-\mathbb{X}^2}}\,d\mathbb{X}\right)$$

$$= 4\pi r\left(\frac{-1}{2}\int_{\mathbb{X}=-r}^{r}\frac{d(r^2-\mathbb{X}^2)}{\sqrt{r^2-\mathbb{X}^2}} + R\int_{\mathbb{X}=-r}^{r}\frac{1}{\sqrt{1-\left(\frac{\mathbb{X}}{r}\right)^2}}\,d\left(\frac{\mathbb{X}}{r}\right)\right)$$

$$= 4\pi r\left(-\sqrt{r^2-\mathbb{X}^2}\Big|_{\mathbb{X}=-r}^{r} + R\sin^{-1}\left(\frac{\mathbb{X}}{r}\right)\Big|_{\mathbb{X}=-r}^{r}\right)$$

$$= 4\pi^2 rR = (2\pi r)\cdot(2\pi R)$$