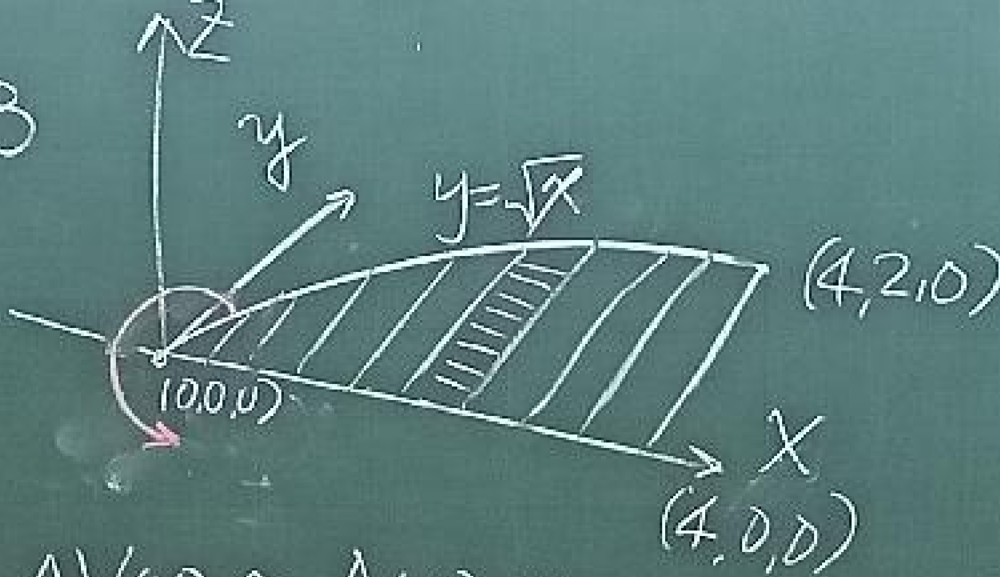
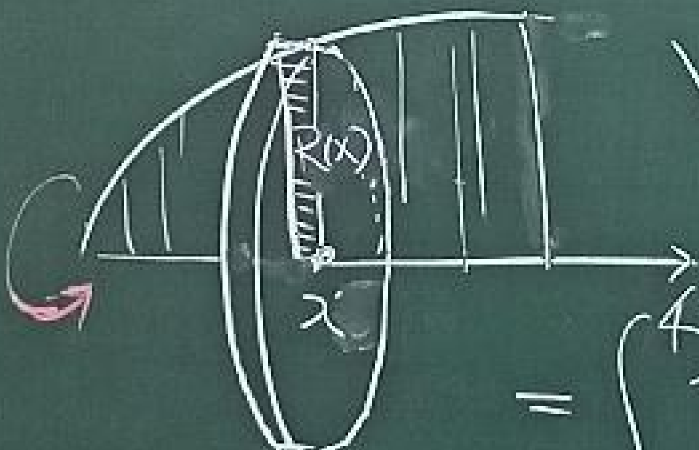
Volume of Revolution

(I). Method of disks

$\boxed{\text{Eg}}3$



$$\Delta V(x) \cong A(x)\,\Delta x$$



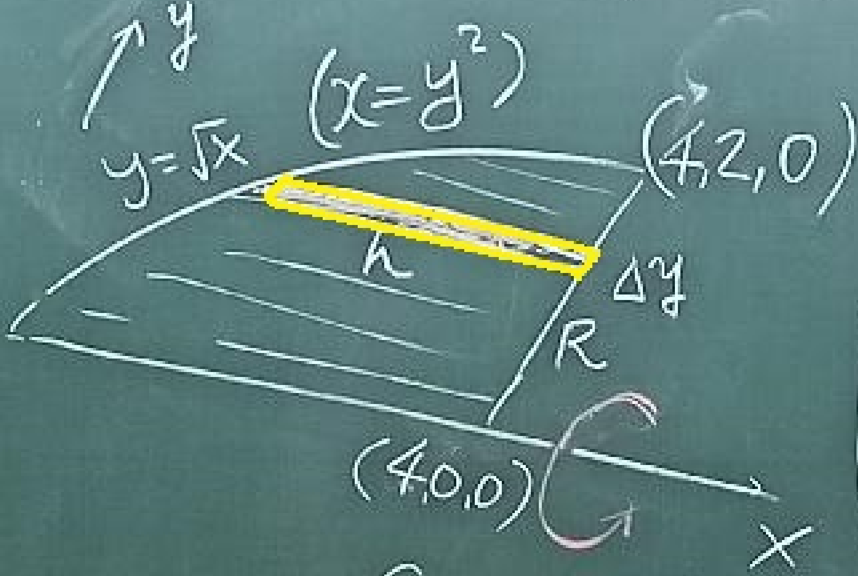
$$V = \int_0^4 A(x)\,dx$$

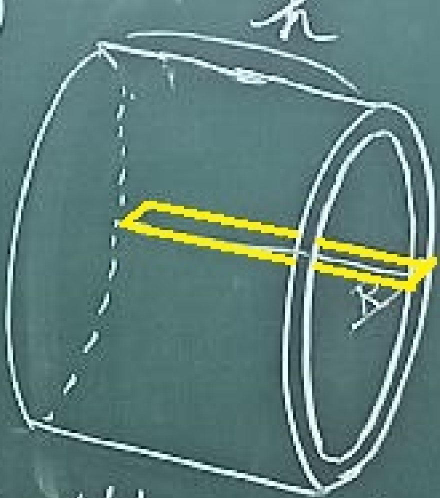
$$= \int_0^4 \pi R(x)^2\,dx$$

Here $R(x) = y = \sqrt{x}$

$$\therefore V = \int_0^4 \pi x\,dx = 8\pi$$

(II) Method of Cylindrical Shells





$$\Delta V = (\text{Surface Area})\cdot(\text{thick}\underline{\text{ness}})$$

$$= 2\pi R(y)\, h(y) \cdot \Delta y$$

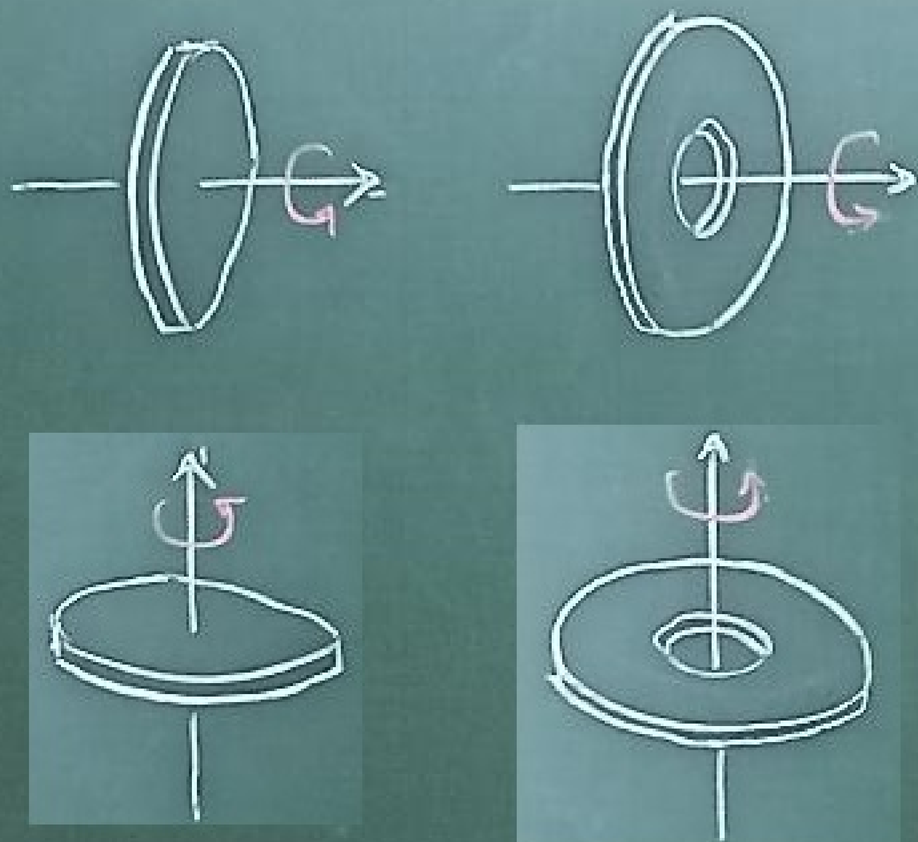
$$= 2\pi y\,(4-y^2)\,\Delta y$$

$$V = \int_0^2 2\pi y\,(4-y^2)\,dy$$

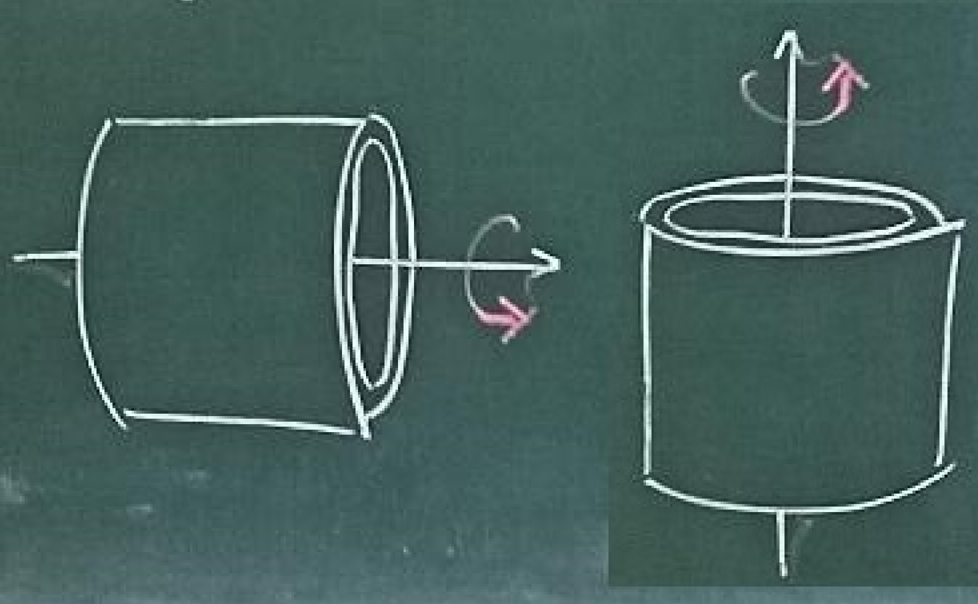
$$= 2\pi\left(2y^2 - \frac{y^4}{4}\right)\Big|_0^2$$

$$= 8\pi$$

Disks (Washers)

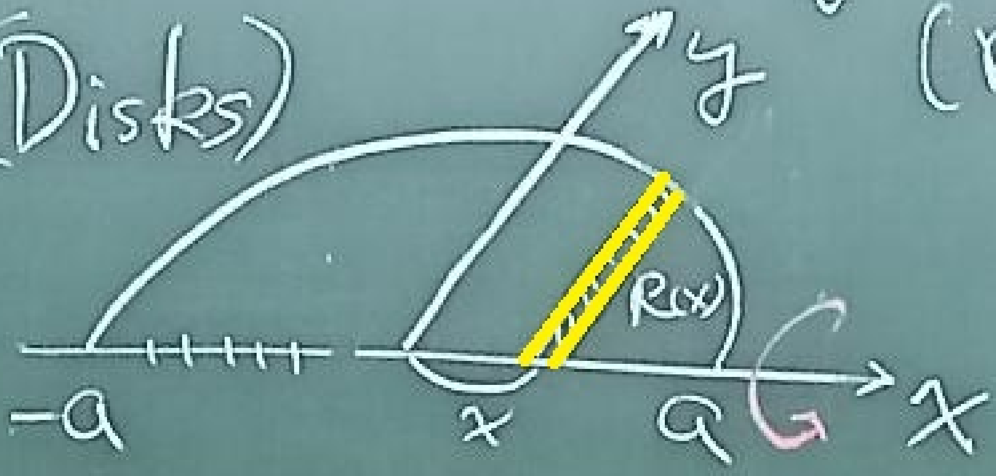


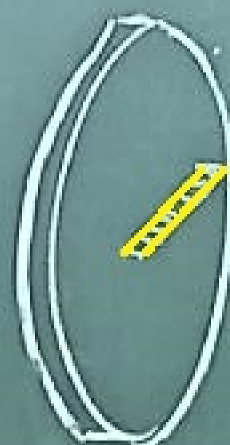
Cylindrical Shells



[Eg] Volume of a ball (radius a)

(Disks)





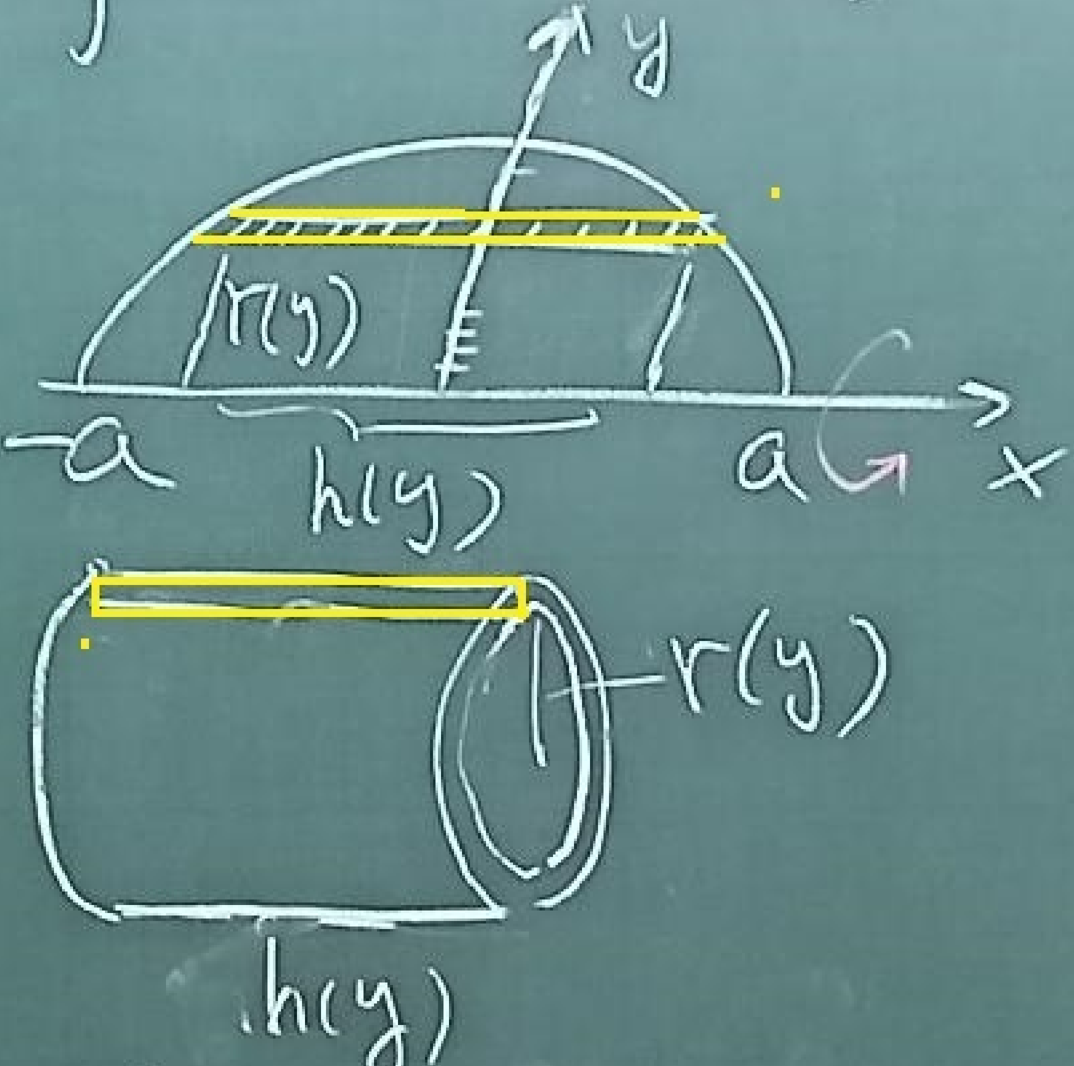
$$V = \int_{-a}^{a} \pi \left(R^2(x) - 0^2 \right) dx$$

$$x^2 + R^2(x) = a^2$$

$$= \int_{-a}^{a} \pi \left(a^2 - x^2 \right) dx$$

$$= \frac{4}{3} \pi a^3$$

Cylindrical Shells



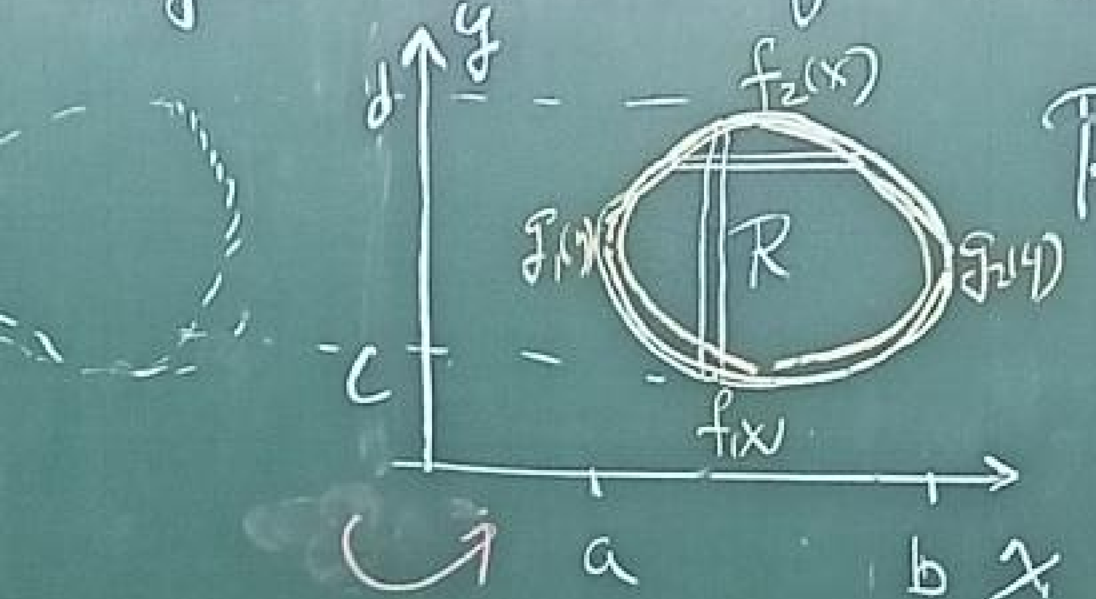
$$V = \int_{y=0}^{a} 2\pi r(y)\, h(y)\, dy$$

$$r(y) = y \,.\; h(y) = 2\sqrt{a^2 - y^2}$$

$$= \pi \int_0^a 4y\sqrt{a^2-y^2}\, dy$$

$$= -2\pi \int_0^a \sqrt{a^2-y^2}\, d(a^2-y^2)$$

Eg2. Volume of a donut



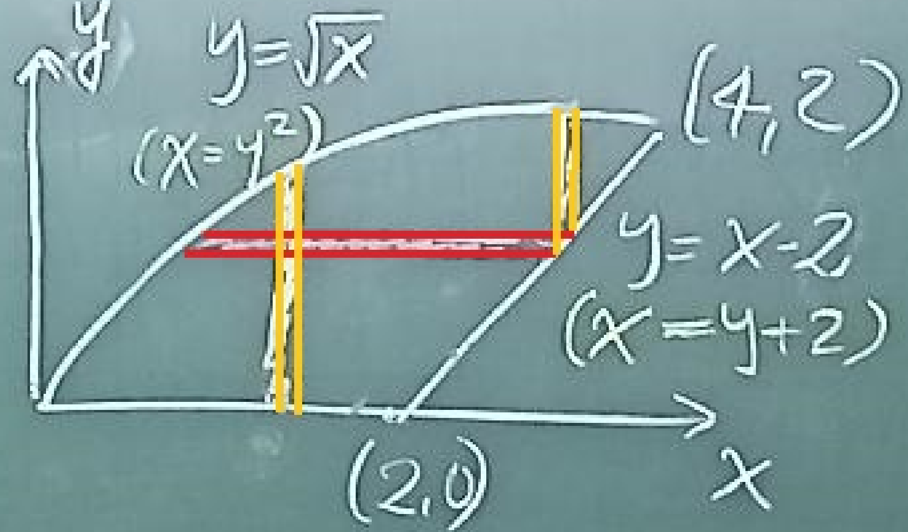
$$R=\begin{cases}0<a\le x\le b\\ f_1(x)\le y\le f_2(x)\end{cases}$$

$$=\begin{cases}c\le y\le d\\ 0<g_1(y)\le x\le g_2(y)\end{cases}$$

$$V_{\circlearrowright} \overset{\text{disks}}{=} \int_{y=c}^{d} \pi\Big(\underset{\text{外圓}}{g_2^2(y)}-\underset{\text{內圓}}{g_1^2(y)}\Big)\underset{\text{厚}}{dy}$$

$$\overset{\text{Shells}}{=} \int_a^b \underbrace{2\pi \underset{\text{半徑}}{x}}_{\text{圓周長}}\Big(\underbrace{f_2(x)-f_1(x)}_{\text{高度}}\Big)\underset{\text{厚}}{dx}$$

Eg 3



$$V \overset{\text{Shells}}{=\!=} \int_0^2 2\pi y\left(y+2-y^2\right)\underline{dy}$$

$$\overset{\text{disk}}{=\!=} \int_0^2 \pi\sqrt{x}^2\,\underline{dx} + \int_2^4 \pi\left(\sqrt{x}^2-(x-2)^2\right)\underline{dx}$$

$$V \overset{\text{Shell}}{=\!=} \int_0^2 2\pi x\left(\sqrt{x}-0\right)\underline{dx}$$

$$+\int_2^4 2\pi x\left(\sqrt{x}-(x-2)\right)\underline{dx}$$

$$\overset{\text{disk}}{=\!=} \int_0^2 \pi\left((y+2)^2-(y^2)^2\right)\underline{dy}$$