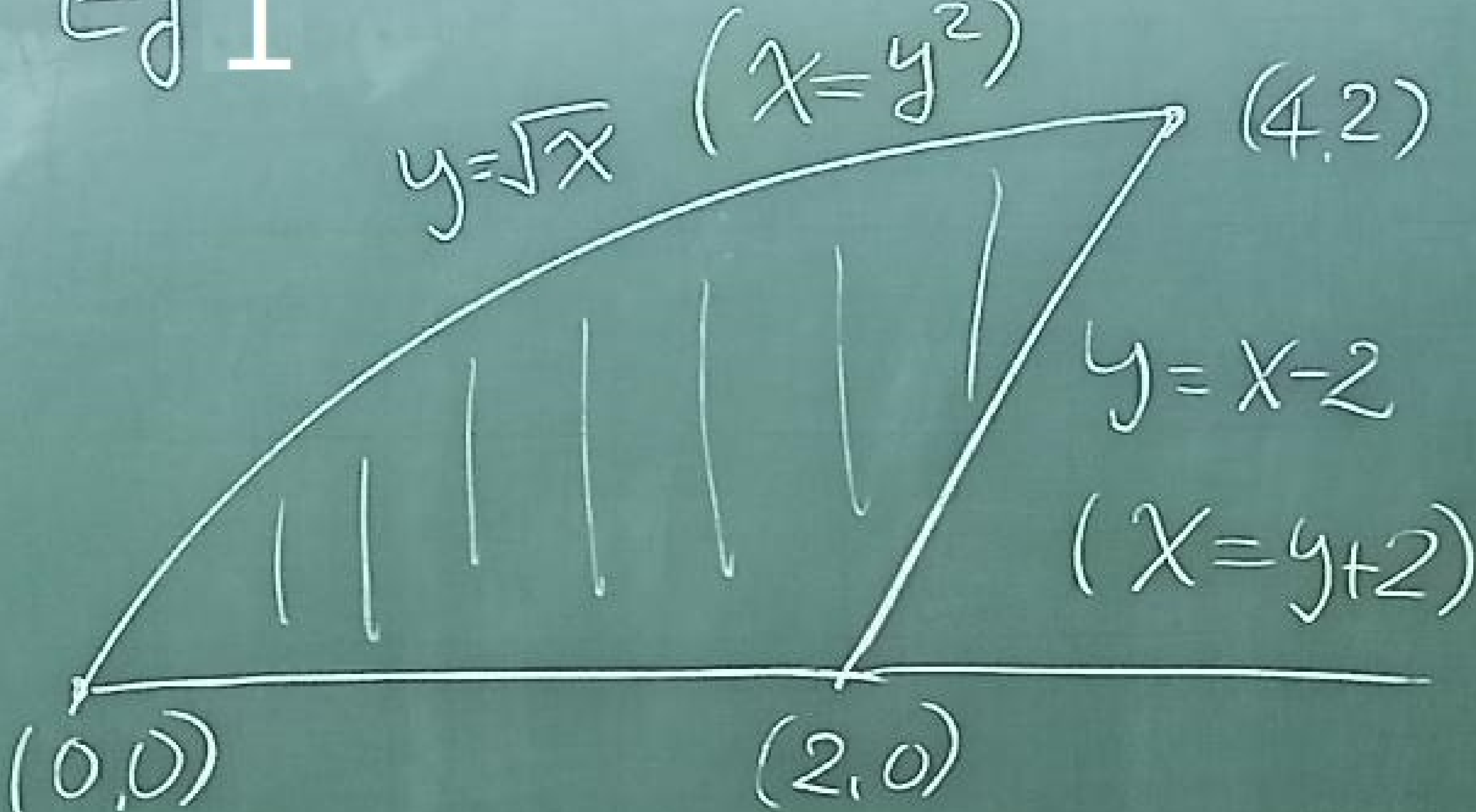
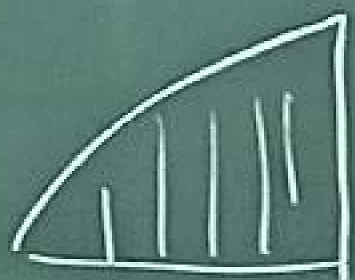
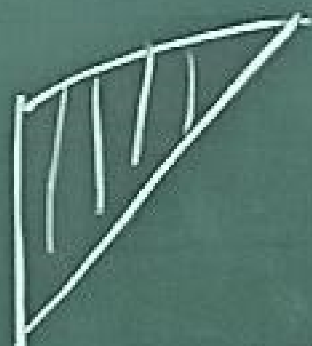
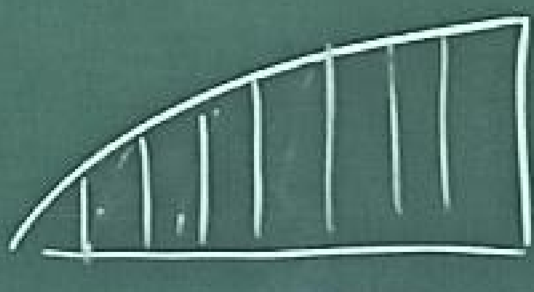
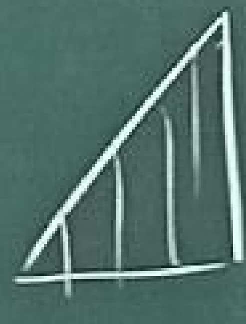
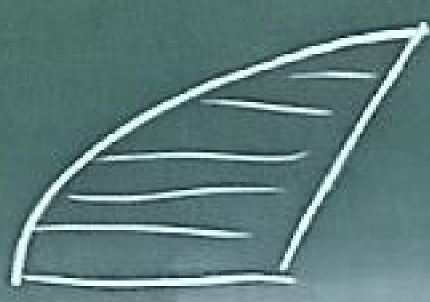
Area between Curves

Eg 1



$A =$  $+$  $(\int dx)$ (I)

$=$  $-$  $(\int dx)$ (II)

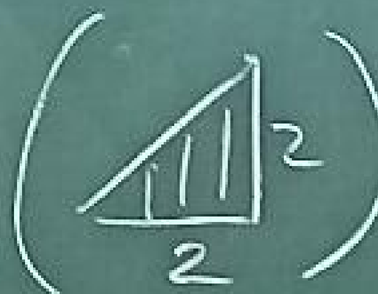
$=$  $(\int dy)$ (III)

Sol:

$$(\mathrm{I}) = \int_0^2 (\underbrace{\sqrt{x} - 0}_{上-下})\,dx + \int_2^4 (\underbrace{\sqrt{x} - (x-2)}_{上-下})\,dx$$

$$(\mathrm{II}) = \int_0^4 (\sqrt{x} - 0)\,dx - \int_2^4 ((x-2) - 0)\,dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} \Big|_0^4 - \frac{1}{2} 2 \cdot 2$$



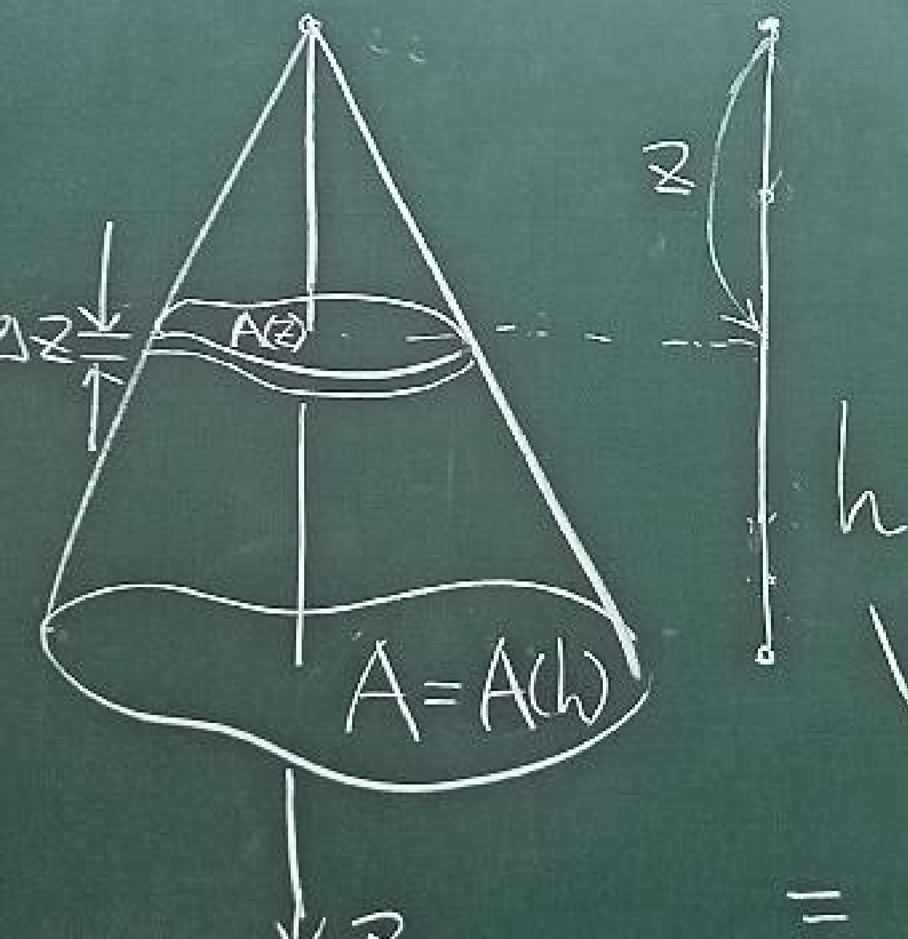
$$= \frac{16}{3} - 2 = \frac{10}{3}$$

$$(\mathrm{III}): \int_{y=0}^{2} (\underbrace{(y+2)}_{右} - \underbrace{y^2}_{左})\,dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_0^2 = \frac{10}{3}$$

Volume by Cross Section

Eg 1



$$\frac{A(z)}{A(h)} = \left(\frac{z}{h}\right)^2$$

$$A(z) = \frac{A}{h^2} z^2$$

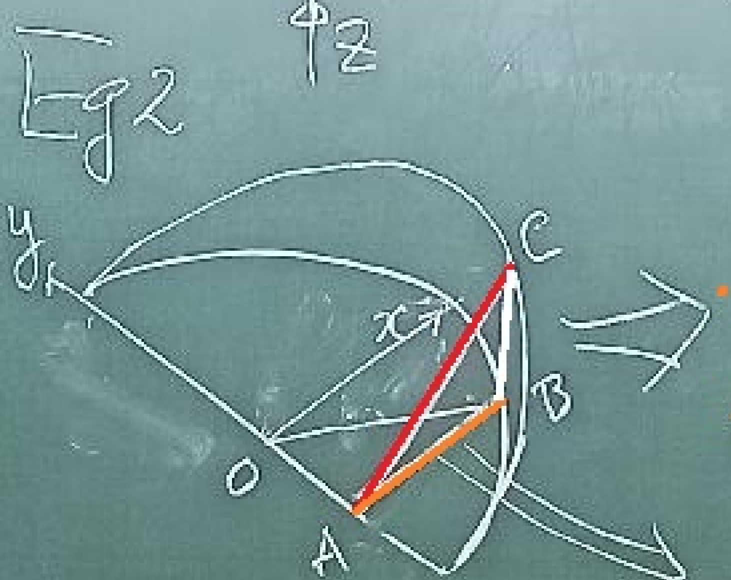
$$\Delta V(z) \cong A(z)\Delta z$$

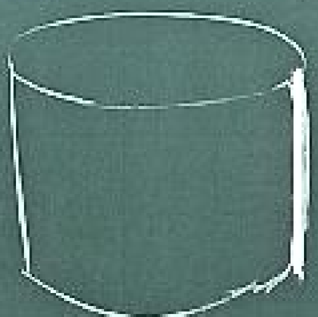
$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(z_k)\Delta z_k$$

$$= \int_0^h A(z)\,dz = \int_0^h \frac{A}{h^2} z^2\,dz$$

$$= \frac{1}{3}Ah$$

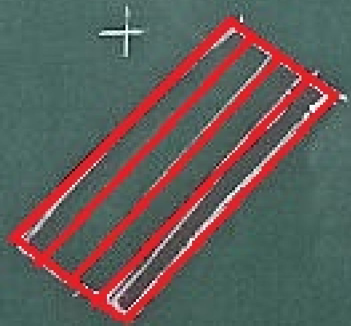
Eg2





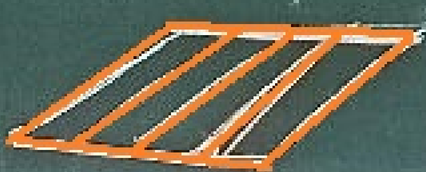
$x^2+y^2=9$

+



$x=z$

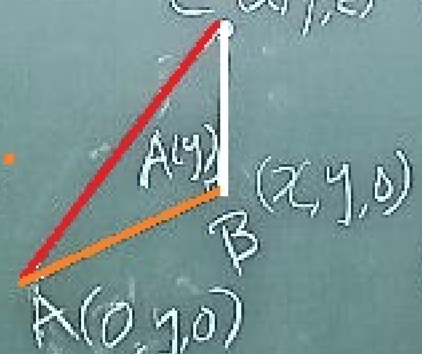
+



$x=0$

Method I: cut along "y = constant"

$$V=\int_{-3}^{3}A(y)\,dy$$



$$A(y)=\frac{1}{2}\overline{AB}\cdot\overline{BC}$$

$$\Rightarrow \overline{AB}=\sqrt{3^2-y^2}$$

$$\overline{BC}=z=x=\overline{AB}$$

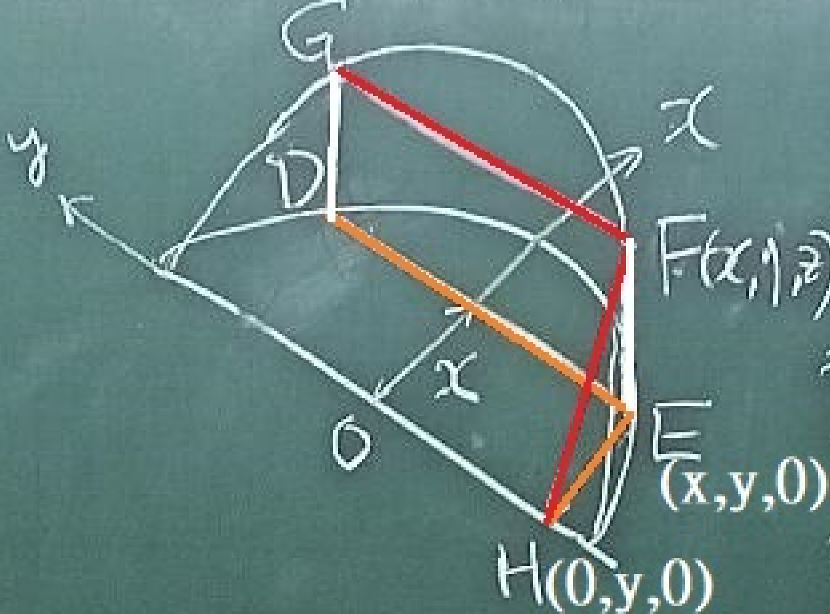
$$\therefore\ \overline{BC}=\overline{AB}=\sqrt{9-y^2}$$

$$A(y)=\frac{1}{2}(9-y^2)$$

$$V=\int_{-3}^{3}\frac{1}{2}(9-y^2)=\frac{9y}{2}-\frac{y^3}{6}\Big|_{-3}^{3}$$

$$=18$$

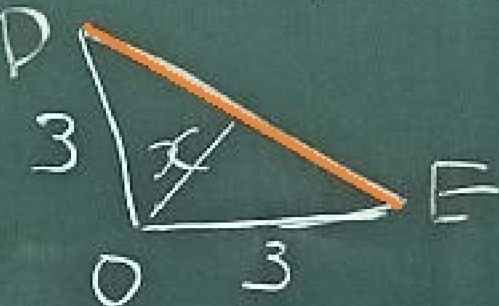
Method II, Cut along "$x=\text{const}$"



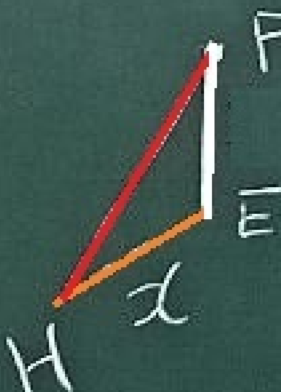
$$A(x)=\overline{DE}\cdot\overline{EF}$$

$$\overline{DE}=2\sqrt{3^2-x^2}$$

$$\overline{EF}=\overline{HE}=x$$



$$V=\int_0^3 A(x)\,dx$$



$$=\int_0^3 2x\sqrt{9-x^2}\,dx$$

$$=\int_0^3 \sqrt{9-x^2}\,dx^2$$

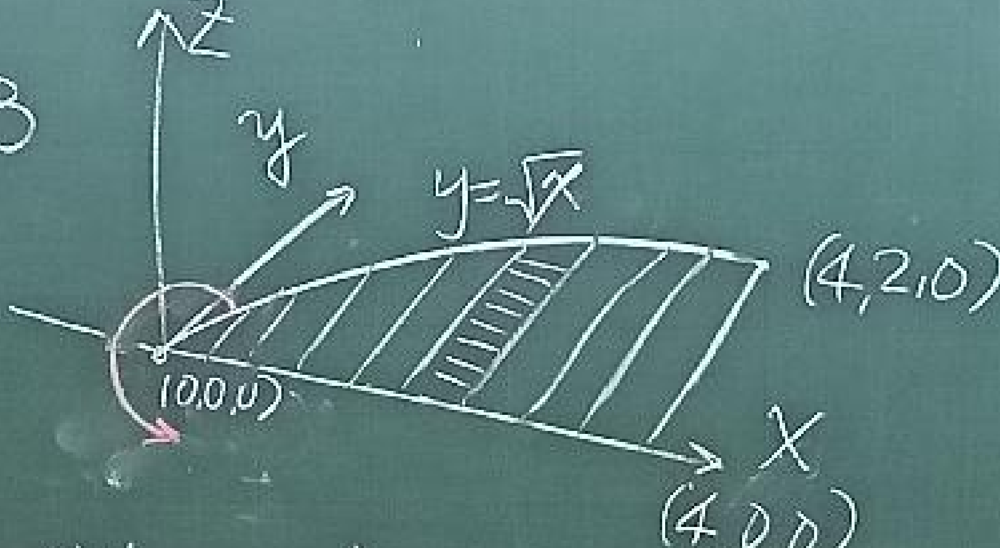
$$=-\int_0^3 \sqrt{9-x^2}\,d(9-x^2)$$

$$=-\frac{2}{3}(9-x^2)^{\frac{3}{2}}\Big|_0^3=18$$

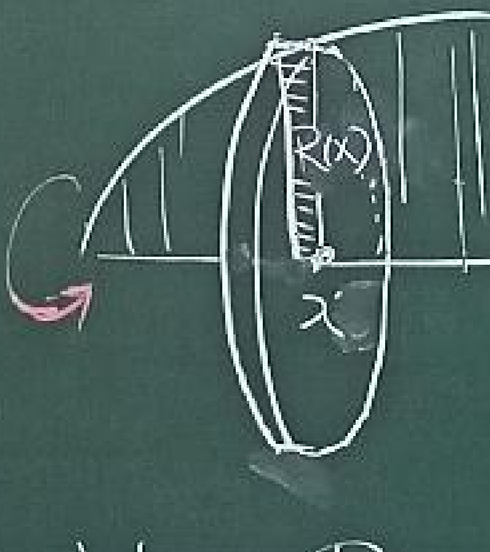
Volume of Revolution

(I). Method of disks

$\boxed{\text{Eg}}\,3$



$$\Delta V(x) \cong A(x)\,\Delta x$$



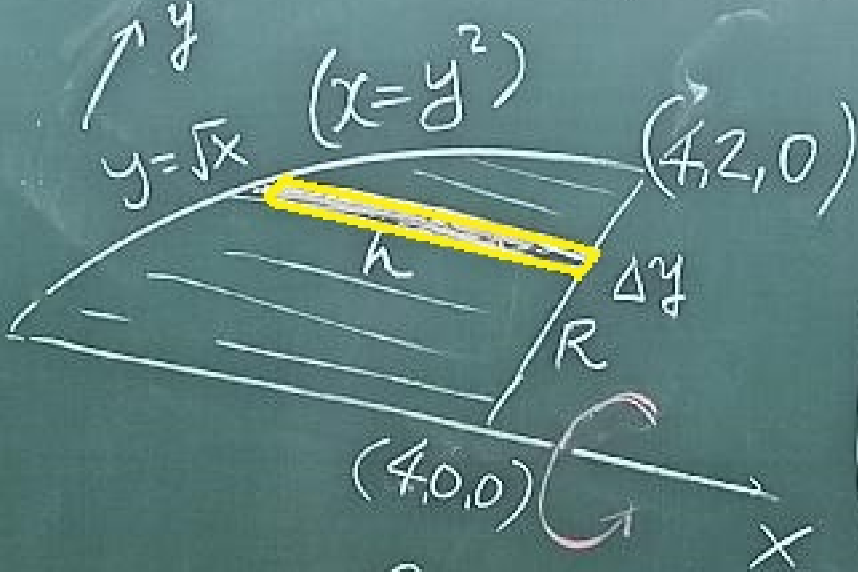
$$V = \int_0^4 A(x)\,dx$$

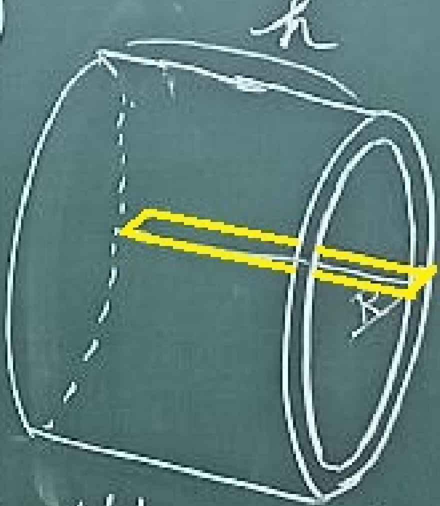
$$= \int_0^4 \pi R(x)^2\,dx$$

Here $R(x) = y = \sqrt{x}$

$$\therefore V = \int_0^4 \pi x\,dx = 8\pi$$

(II) Method of Cylindrical Shells





$$\Delta V = (\text{Surface Area}) \cdot (\text{thickness})$$

$$= 2\pi R(y)\, h(y) \cdot \Delta y$$

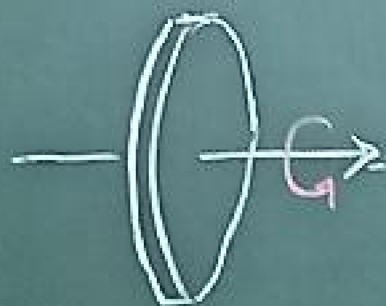
$$= 2\pi y (4 - y^2)\, \Delta y$$

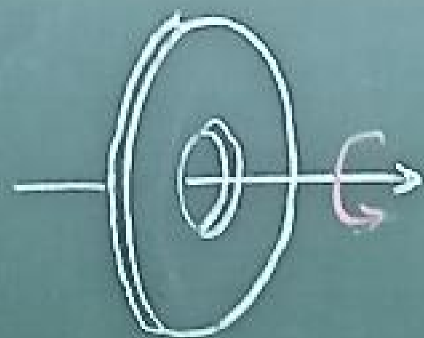
$$V = \int_0^2 2\pi y (4 - y^2)\, dy$$

$$= 2\pi \left(2y^2 - \frac{y^4}{4} \right) \Big|_0^2$$

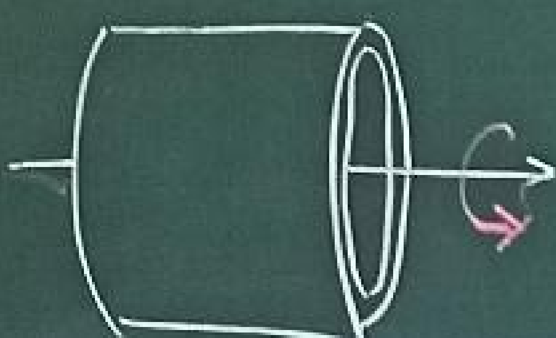
$$= 8\pi$$

Disks (Washers)



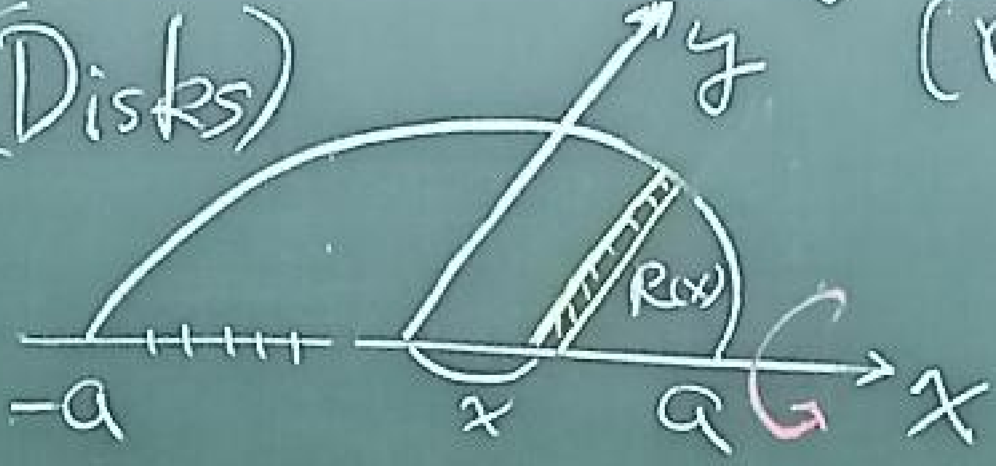


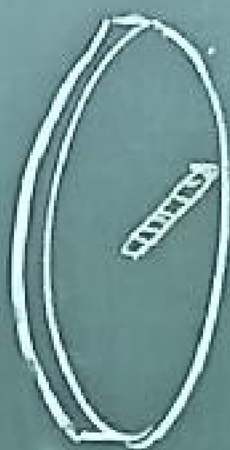
Cylindrical Shells



[Eg] Volume of a ball

(Disks) (radius a)





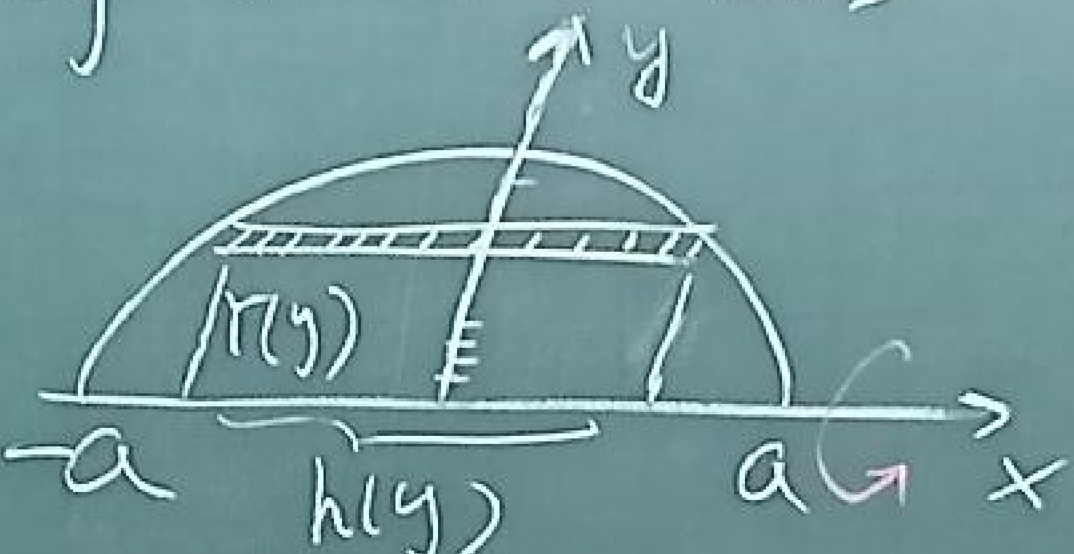
$$V=\int_{-a}^{a}\pi\left(R^2(x)-0^2\right)dx$$

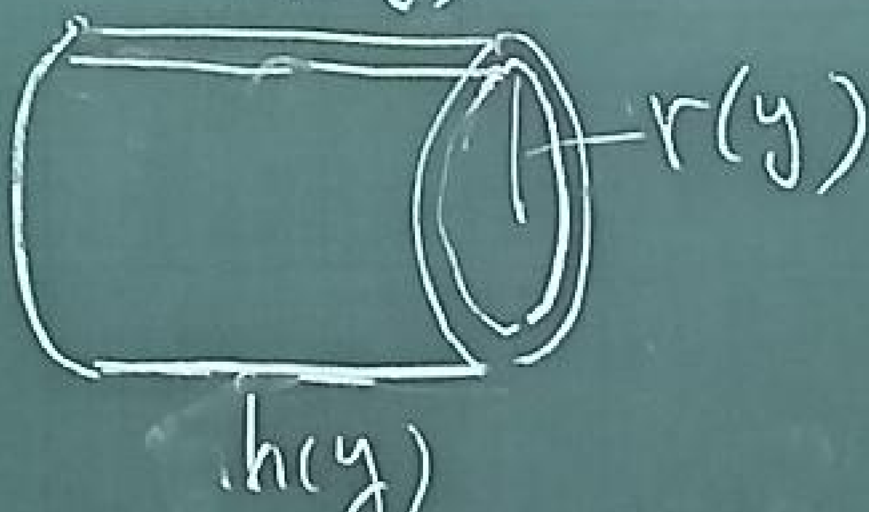
$$x^2+R^2(x)=a^2$$

$$=\int_{-a}^{a}\pi\left(a^2-x^2\right)dx$$

$$=\frac{4}{3}\pi a^3$$

Cylindrical Shells





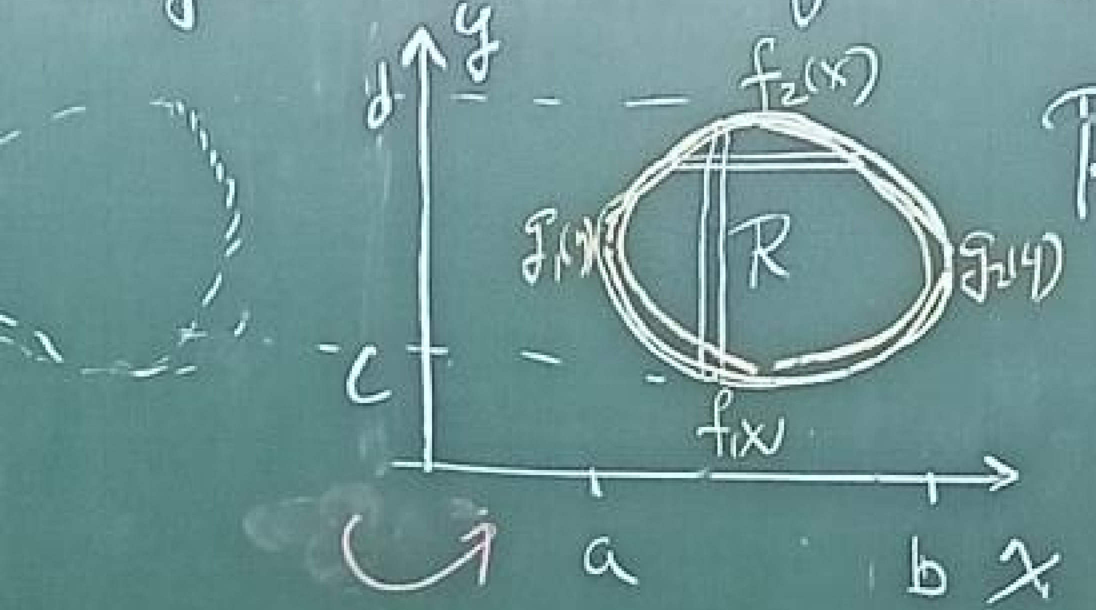
$$V = \int_{y=0}^{a} 2\pi r(y)\, h(y)\, dy$$

$$r(y) = y \,.\, h(y) = 2\sqrt{a^2 - y^2}$$

$$= \pi \int_0^a 2y\sqrt{a^2-y^2}\, dy$$

$$= -\pi \int_0^a \sqrt{a^2-y^2}\, d(a^2-y^2)$$

Eg2. Volume of a donut

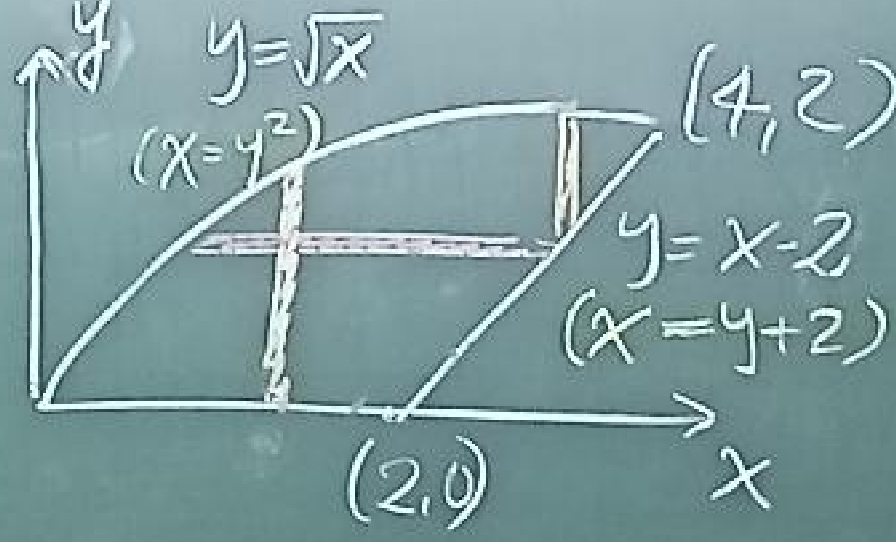


$$R=\begin{Bmatrix} 0<a\le x\le b \\ f_1(x)\le y\le f_2(x) \end{Bmatrix} = \begin{Bmatrix} c\le y\le d \\ 0<g_1(y)\le x\le g_2(y) \end{Bmatrix}$$

$$V \overset{\text{disks}}{=} \int_{y=c}^{d} \pi\left(\underset{\text{外圓}}{g_2^2(y)}-\underset{\text{內圓}}{g_1^2(y)}\right)\underset{\text{厚}}{dy}$$

$$\overset{\text{Shells}}{=} \int_a^b \underbrace{2\pi \underset{\text{半徑}}{x}}_{\text{圓周長}}\underbrace{\left(f_2(x)-f_1(x)\right)}_{\text{高度}}\underset{\text{厚}}{dx}$$

Eg3



$$V \overset{\text{Shells}}{=\!=} \int_0^2 2\pi y\left(y+2-y^2\right)dy$$

$$\overset{\text{disk}}{=\!=} \int_0^2 \pi\sqrt{x}^2\,dx + \int_2^4 \pi\left(\sqrt{x}^2-(x-2)^2\right)dx$$

$$V \overset{\text{Shell}}{=\!=} \int_0^2 2\pi x\left(\sqrt{x}-0\right)dx$$

$$+\int_2^4 2\pi x\left(\sqrt{x}-(x-2)\right)dx$$

$$\overset{\text{disk}}{=\!=} \int_0^2 \pi\left((y+2)^2-(y^2)^2\right)dy$$