The Substitution Method [Thm 6, 7]

Thm 6 If $u = g(x)$ is differentiable, $f(u)$ is cont. on range of $g(x)$ and $F'(u) = f(u)$, then

$$\boxed{\text{Substitute } g'(x)dx = \frac{du}{\cancel{dx}}\cancel{dx} = du}$$

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C$$

$$\left(= \int f(g(x))\,dg(x) = F(g(x)) + C\right)$$

Thm 7 If g' is cont. on $[a,b]$, $f(u)$ is cont. on range of $g(x) = u$, and $F'(u) = f(u)$, then

$$\int_{x=a}^{b} f(g(x))g'(x)\,dx = \int_{u=g(a)}^{g(b)} f(u)\,du = F(u)\Big|_{u=g(a)}^{g(b)}$$

$$\left(= \int_{x=a}^{b} f(g(x))\,dg(x) = F(g(x))\Big|_{x=a}^{b}\right)$$

Eg1 $\displaystyle\int_0^1 \underbrace{(x^3+x)^5}_{g(x)^5} \underbrace{(3x^2+1)}_{g'(x)}\,dx$

$(\text{or } u^5 \quad , \quad u')$

$$= \int_0^1 (x^3+x)^5 \, d(x^3+x)$$

$$= \int_{x=0}^1 u^5\,du = \int_{x=0}^1 \frac{d}{du}\left(\frac{u^6}{6}\right)du$$

$$= \frac{1}{6}(x^3+x)^6 \Big|_{x=0}^{1}$$

$$= \frac{1}{6}u^6 \Big|_{x=0}^{1}$$

$$= \frac{32}{3}$$

$$= \frac{1}{6}u^6 \Big|_{u=g(0)=1}^{g(1)=2}$$

$$= \frac{1}{6}(64-0) = \frac{32}{3}$$

$$\left(\text{or} = \int_{x=0}^1 \frac{d}{dx}\left(\frac{1}{6}g(x)^6\right)dx \right.$$

$$= \frac{1}{6}g(x)^6 \Big|_{x=0}^{1}$$

$$\left. = \frac{32}{3} \right)$$

Eg 2: $\int_0^1 \sqrt{2x+1}\,dx$

Sol Let $u=2x+1$, $\frac{du}{dx}=2$

$$= \int_{x=0}^{1} \frac{1}{2}u^{\frac{1}{2}} \cdot \underbrace{2\,dx}_{=\frac{du}{\cancel{dx}}\cancel{dx} = du}$$

$$= \int_{u=1}^{3} \frac{d}{du}\left(\frac{1}{3}u^{\frac{3}{2}}\right) du$$

$$= \frac{1}{3}u^{\frac{3}{2}}\Big|_{u=1}^{3}$$

$$= \frac{1}{3}\left(3\sqrt{3}-1\right)$$

$$= \int_0^1 (2x+1)^{\frac{1}{2}} \cdot \frac{1}{2} \cdot \frac{d(2x+1)}{\cancel{dx}}\cancel{dx}$$

$$= \int_{x=0}^{1} \frac{1}{2}(2x+1)^{\frac{1}{2}}\, d(2x+1)$$

$$= \frac{1}{3}(2x+1)^{\frac{3}{2}}\Big|_{x=0}^{1}$$

$$= \frac{1}{3}\left(3\sqrt{3}-1\right)$$

Other examples:

$$\int f(\sin x)\cos x\,dx = \int f(\sin x)\,d\sin x = F(\sin x) + C$$

$$\int f(\cos x)\sin x\,dx = \int f(\cos x)\,d(-\cos x) = -F(\cos x) + C$$

$$\int e^{g(x)} g'(x)\,dx = \int e^{g(x)}\,dg(x) = e^{g(x)} + C$$

$$\int \frac{g'(x)}{g(x)}\,dx = \int \frac{1}{g(x)}\,dg(x) = \ln|g(x)| + C$$

Eg 3 $\int \sec^2(7x+3)\,dx$

$$= \int \sec^2(7x+3)\,\frac{1}{7}\,d(7x+3)$$

$$= \frac{1}{7}\tan(7x+3) + C$$

Eg 4 $\int_0^1 x^2 e^{x^3}\,dx$

$$= \int_0^1 e^{x^3}\cdot\frac{1}{3}\,dx^3$$

$$= \frac{1}{3}e^{x^3}\Big|_0^1 = \frac{e-1}{3}$$

Eg 5 $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,dx$

$$= \int \frac{d\sin x}{1+\sin^2 x}$$

$$= \tan^{-1}(\sin x) + C.$$

Eg 6 $\int \sin^3 x \, dx$

$= \int \sin^2 x \sin x \, dx$

$= -\int (1-\cos^2 x) \, d\cos x$

$= -\int (1-c^2) \, dc$

$= -c + \frac{c^3}{3} + C$

$= -\cos x + \frac{\cos^3 x}{3} + C$

Eg 7 $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$

$$= \int \frac{d(z^2+1)}{(z^2+1)^{\frac{1}{3}}}$$

$$= \int (z^2+1)^{\frac{1}{3}}\, d(z^2+1)$$

$$= \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C$$

Eg 8 $\int \sin^2 x\, dx$

$$= \int \frac{1-\cos 2x}{2}\, dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Eg 9 $\displaystyle\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\,\sin\sqrt{\theta}}\,d\theta$

$$= \int \frac{\cos\sqrt{\theta}}{\sin\sqrt{\theta}}\,\frac{d\theta}{\sqrt{\theta}}$$

$$\theta^{-\frac{1}{2}}\,d\theta = 2\,d\,\theta^{\frac{1}{2}}$$

$$= 2\int \frac{\cos\sqrt{\theta}}{\sin\sqrt{\theta}}\,d\sqrt{\theta}$$

$$\cos\sqrt{\theta}\,d\sqrt{\theta} = d\sin\sqrt{\theta}$$

$$= 2\int \frac{d\sin\sqrt{\theta}}{\sin\sqrt{\theta}}$$

$$= \ln|\sin\sqrt{\theta}| + C$$

Eg 10 $\int \sec x \, dx$

$$= \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx$$

$$= \int \frac{d\sin x}{1-\sin^2 x} = \int \frac{ds}{1-s^2}$$

$$= \frac{1}{2}\int \left(\frac{1}{1+s} + \frac{1}{1-s}\right) ds$$

$$= \frac{1}{2}\int \frac{d(1+s)}{1+s} - \frac{1}{2}\int \frac{d(1-s)}{1-s}$$

$$= \frac{1}{2}\ln|1+s| - \frac{1}{2}\ln|1-s| + C$$

$$= \frac{1}{2}\ln\left|\frac{1+s}{1-s}\right| + C$$

$s = \sin x \quad -1 \leqq s \leqq 1$

$\therefore |1+s| = 1+s,\ |1-s| = 1-s$

$$= \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + C$$

$$= \ln\sqrt{\frac{1+\sin x}{1-\sin x}} + C$$

Eg 11 $\int \frac{1}{e^{x}+e^{-x}}dx = \int \frac{1}{e^{2x}+1}e^{x}dx$

$$= \int \frac{1}{1+e^{-2x}}e^{-x}dx$$

$$= \int \frac{1}{1+(e^{-x})^{2}}d(-e^{-x})$$

$$= \underbrace{-\tan^{-1}(e^{-x})+C}_{(\mathrm{II})}$$

$$= \int \frac{1}{1+(e^{x})^{2}}de^{x}$$

$$= \underbrace{\tan^{-1}(e^{x})+C}_{(\mathrm{I})}$$

Rm $\tan^{-1}\left(\frac{b}{a}\right)+\tan^{-1}\left(\frac{a}{b}\right)=\frac{\pi}{2}$ $(a>0,\ b>0)$

$\Rightarrow (\mathrm{I})=(\mathrm{II})$

Eg12 $\int x\sqrt{2x+1}\,dx$

let $u=\sqrt{2x+1}$

$u^2=2x+1 \Rightarrow 2u\,du=2\,dx$

(i.e. $2u\frac{du}{dx}=2$)

$$=\int \overbrace{\frac{u^2-1}{2}}^{x}\,\overbrace{u}^{\sqrt{2x+1}}\cdot(\overbrace{u\,du}^{dx})$$

$$=\frac{u^5}{10}-\frac{u^3}{6}+C=\frac{(2x+1)^{\frac{5}{2}}}{10}-\frac{(2x+1)^{\frac{3}{2}}}{6}+C$$

<u>or</u> let $v=2x+1$, $dv=2\,dx$

$$=\int\frac{v-1}{2}\sqrt{v}\,\frac{dv}{2}=\frac{v^{\frac{5}{2}}}{10}-\frac{v^{\frac{3}{2}}}{6}+C$$

= same.

Eg 13 $\int \sqrt{\frac{x-1}{x^5}}\, dx,\ x>0$

$\frac{dx}{\sqrt{x}} = 2d\sqrt{x}$ (X)

$\frac{dx}{x} = d\ln x$ (X)

$\frac{dx}{x^{\frac{3}{2}}} = -2dx^{-\frac{1}{2}}$ (X)

$\frac{dx}{x^2} = -dx^{-1}$ (O)

$= \int \sqrt{\frac{x-1}{x}}\ \frac{dx}{x^2}$

$= -\int \sqrt{1-x^{-1}}\, dx^{-1}$

$= \int \sqrt{1-x^{-1}}\, d(1-x^{-1}) = \frac{2}{3}(1-x^{-1})^{\frac{3}{2}} + C$

Eg14 $\int (x+5)^2 (x-5)^{\frac{1}{3}} dx$

Let $x-5=y$. $x+5=y+10$

$= \int (y+10)^2 y^{\frac{1}{3}} dy$

$= \int y^{\frac{7}{3}} + 20y^{\frac{4}{3}} + 100y^{\frac{1}{3}} dy$

$= \frac{3}{10} y^{\frac{10}{3}} + \frac{60}{7} y^{\frac{7}{3}} + 75 y^{\frac{4}{3}} + C$

$= \frac{3}{10}(x-5)^{\frac{10}{3}} + \frac{60}{7}(x-5)^{\frac{7}{3}} + 75(x-5)^{\frac{4}{3}} + C$