Fundamental Theorem of Calculus

Part I: If f is cont. on $[a,b]$ then $F(x) \overset{\text{def}}{=\!=} \int_a^x f(t)\,dt$ is cont. on $[a,b]$, differentiable on (a,b) and

$$F'(x) = f(x)$$

i.e. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Part II If f is continuous on $[a,b]$

~~If~~ and G is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = G(b)-G(a)$$

i.e. $G(b)-G(a) = \int_a^b G'(x)\,dx$.

Rm The textbook use F here.

Rm $\int_a^b f(x)\,dx$: definite integral.

$\int f(x)\,dx \equiv \int^x f(t)\,dt$: antiderivative of f (indefinite integral)

Part II (Strong Version)

Let G be any antiderivative of f. If f is cont. on (a,b) and G is cont. on $[a,b]$, then

$$G(b)-G(a)=\lim_{\substack{\alpha\to a^+\\ \beta\to b^-}}\int_\alpha^\beta f(t)\,dt$$

(Improper integral
See Section 8.8)

<u>Pf</u> Standard FTC II $\Rightarrow G(\beta)-G(\alpha)=\int_\alpha^\beta f(t)\,dt$

Take $\lim_{\substack{\alpha\to a^+\\ \beta\to b^-}}$ + continuity of G at $x=a,b$.

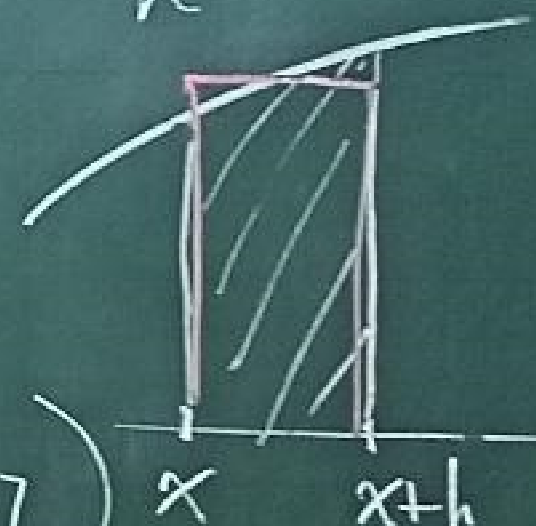
<u>Eg</u>: $\int_0^1 x^{-\frac{1}{3}}\,dx \overset{\text{def}}{=} \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 x^{-\frac{1}{3}}\,dx$

$\overset{\left(\substack{\text{FTC II}\\ \text{strong version}}\right)}{=} \frac{3}{2}x^{\frac{2}{3}}\Big|_0^1 = \frac{3}{2}$

Pf (part I)

$$\frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0} \frac{\int_a^{x+h}(*) - \int_a^{x}(*)}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$



$$= \lim_{h\to 0}\Big(\underbrace{\text{average of } f \text{ on } [x, x+h]}_{(**)}\Big)$$

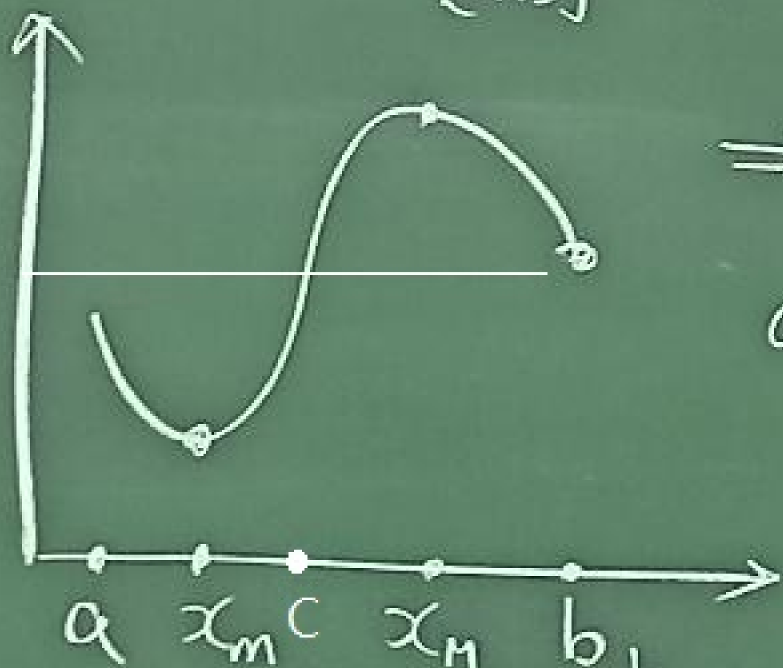
$$\Rightarrow \min_{[x,x+h]} f \le (**) \le \max_{[x,x+h]} f$$

$$\xrightarrow{\text{Thm 3}} (**) = f(c),\ c\in[x, x+h]$$

$$\therefore \frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0} f(c) = f(x)$$

Thm 3. If f is cont. on $[a,b]$ then $f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx$ for some $c\in[a,b]$

pf Suppose $x_m, x_M\in[a,b]$ with $f(x_m)=\min_{[a,b]} f$, $f(x_M)=\max_{[a,b]} f$



$\Rightarrow [x_m, x_M]\subseteq[a,b]$ or $[x_M, x_m]\subseteq[a,b]$

$$f(x_m)=\min_{[a,b]} f\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\max_{[a,b]} f=f(x_M)$$

Apply I.V.T to f on $[x_m, x_M]$ (or $[x_M, x_m]$)

$$\Rightarrow \exists\, c\in\underset{\text{or }[x_M,x_m]}{[x_m, x_M]}\subseteq[a,b],\ f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx$$

Part (II) proof:

Let $F(x) = \int_a^x f(t)\,dt$

$$\xrightarrow{\text{Part I}} F'(x) = f(x) = G'(x)$$

$$(F-G)' = 0 \Rightarrow G(x) = F(x) + C$$

$$\therefore G(b) - G(a) = (F(b)+C) - (F(a)+C)$$

$$= F(b) - F(a) = \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{\overset{\shortparallel}{0}}$$

$$= \int_a^b f(x)\,dx$$

Examples for Part (I)

Eg4 $\frac{d}{dx}\int_a^x (t^3+1)\,dt = x^3+1$

Eg5 $\frac{d}{dx}\int_x^0 \sin(t^2)\,dt$

$= \frac{d}{dx}\left(-\int_0^x \sin(t^2)\,dt\right)$

$= -\sin(x^2)$

Eg6 $\frac{d}{dx}\int_1^{x^2} e^{\frac{1}{t}}\,dt = \frac{d}{dx}F(x^2)$

$= \frac{d}{du}F(u)\Big|_{u=x^2} \cdot \frac{d}{dx}x^2$

$= e^{\frac{1}{x^2}} \cdot 2x$

Eg 7 $\frac{d}{dx}\int_{x^2}^{x^3} \frac{1}{t^2+1}\,dt$

$$= \frac{d}{dx}\left(\int_0^{x^3} - \int_0^{x^2}\right)$$

$$= \left(\frac{1}{t^2+1}\right)_{t=x^3} \cdot 3x^2 - \left(\frac{1}{t^2+1}\right)_{t=x^2} \cdot 2x$$

$$= \frac{3x^2}{x^6+1} - \frac{2x}{x^4+1}$$

Examples for part II

$$\int_a^b G'(x)\,dx = G(b) - G(a) = G(x)\Big|_a^b$$

Eg 8: $\int_0^{\pi} \cos x\,dx = \sin x\Big|_0^{\pi}$

$$= \sin\pi - \sin 0 = 0$$

Eg 9 $\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx$

$= \int_1^4 \left(\frac{3}{2}x^{\frac{1}{2}} - 4x^{-2}\right)dx$

$= \left(x^{\frac{3}{2}} + 4x^{-1}\right)\Big|_1^4$

$= 9 - 5 = 4$

Eg 10 $\int_0^1 \frac{1}{x^2+1}\,dx$

$= \tan^{-1}x \Big|_0^1 = \frac{\pi}{4} - 0$

Eg 11 $\int_{-1}^1 \sqrt{1-x^2}\,dx =$ 

$= \frac{\pi}{2}$