

Fundamental Theorem of Calculus

Part I: If f is cont. on $[a, b]$

then $F(x) \stackrel{\text{def}}{=} \int_a^x f(t) dt$ is
cont. on $[a, b]$, differentiable
on (a, b) and

$$F'(x) = f(x)$$

$$\text{i.e. } \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

Part II.

If G is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = G(b) - G(a)$$

$$\text{i.e. } G(b) - G(a) = \int_a^b G'(x) dx.$$

Rm The textbook use F here.

Rm $\int_a^b f(x) dx$: definite integral.

$\int f(x) dx \equiv \int^x f(t) dt$: Antiderivative of f
(indefinite integral)

pf (part I)

$$\frac{d}{dx} \int_a^x f(t) dt = \lim_{h \rightarrow 0} \frac{\int_a^{x+h} f(t) dt - \int_a^x f(t) dt}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\int_x^{x+h} f(t) dt}{h}$$

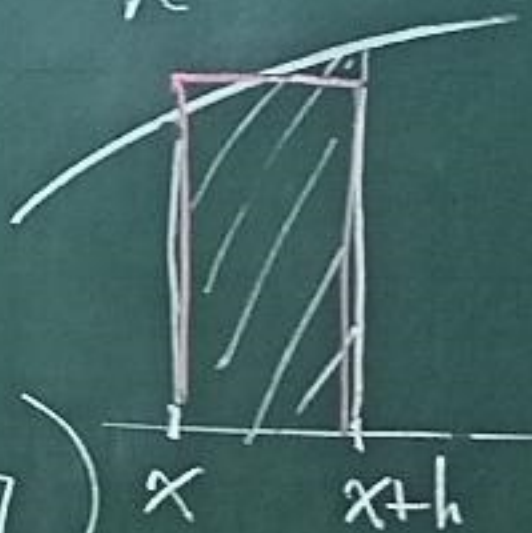
$$= \lim_{h \rightarrow 0} \left(\underbrace{\text{Average of } f \text{ on } [x, x+h]}_{(**)} \right)$$

$$\Rightarrow \min_{[x, x+h]} f \leq (**) \leq \max_{[x, x+h]} f$$

Thm 3

$$\implies (**) = f(c), \quad c \in [x, x+h]$$

$$\therefore \frac{d}{dx} \int_a^x f(t) dt = \lim_{h \rightarrow 0} f(c) = f(x)$$

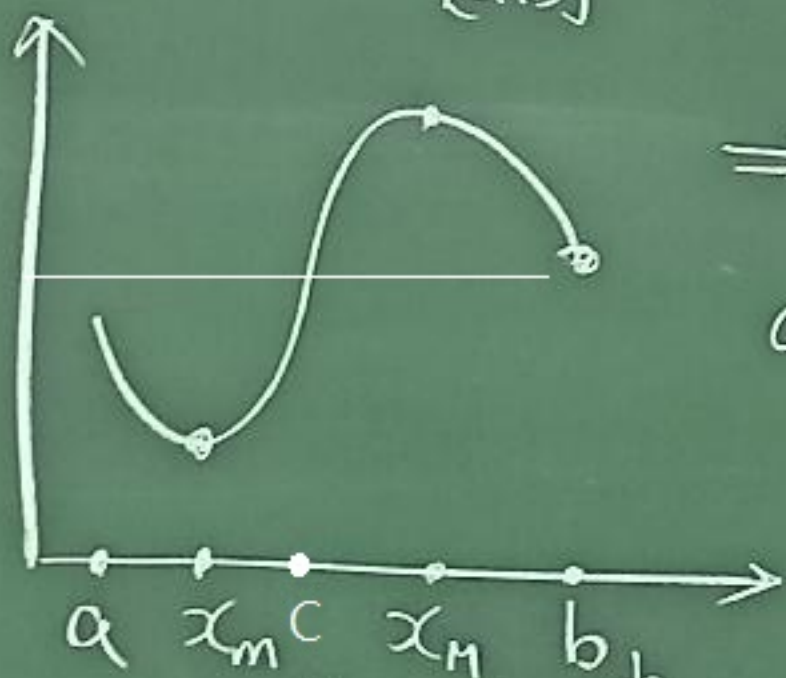


Thm 3. If f is cont. on $[a, b]$

then $f(c) = \frac{1}{b-a} \int_a^b f(x) dx$ for some $c \in [a, b]$

pf Suppose $x_m, x_M \in [a, b]$

with $f(x_m) = \min_{[a, b]} f$, $f(x_M) = \max_{[a, b]} f$



$\Rightarrow [x_m, x_M] \subseteq [a, b]$
or $[x_M, x_m] \subseteq [a, b]$

$$f(x_m) = \min_{[a, b]} f \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \max_{[a, b]} f = f(x_M)$$

Apply I.V.T to f on $[x_m, x_M]$ (or $[x_M, x_m]$)

$$\Rightarrow \exists c \in [x_m, x_M] \subseteq [a, b], \quad f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

Part (II) proof:

$$\text{Let } F(x) = \int_a^x f(t) dt$$

$$\text{Part I} \implies F'(x) = f(x) = G'(x)$$

$$(F-G)' = 0 \implies G(x) = F(x) + C$$

$$\begin{aligned} \therefore G(b) - G(a) &= (F(b) + C) - (F(a) + C) \\ &= F(b) - F(a) = \int_a^b f(t) dt - \underbrace{\int_a^a f(t) dt}_0 \\ &= \int_a^b f(x) dx \end{aligned}$$

Examples for Part (I)

$$\text{Eg 4 } \frac{d}{dx} \int_a^x (t^3 + 1) dt = x^3 + 1$$

$$\text{Eg 5 } \frac{d}{dx} \int_x^0 \sin(t^2) dt$$

$$= \frac{d}{dx} \left(- \int_0^x \sin(t^2) dt \right)$$

$$= - \sin(x^2)$$

$$\text{Eg 6 } \frac{d}{dx} \int_1^{x^2} e^{\frac{1}{t}} dt = \frac{d}{dx} F(x^2)$$

$$= \frac{d}{du} F(u) \Big|_{u=x^2} \cdot \frac{d}{dx} x^2$$

$$= e^{\frac{1}{x^2}} \cdot 2x$$

$$\text{Eq 7 } \frac{d}{dx} \int_{x^2}^{x^3} \frac{1}{t^2+1} dt$$

$$= \frac{d}{dx} \left(\int_0^{x^3} - \int_0^{x^2} \right)$$

$$= \left(\frac{1}{t^2+1} \right)_{t=x^3} \cdot 3x^2 - \left(\frac{1}{t^2+1} \right)_{t=x^2} \cdot 2x$$

$$= \frac{3x^2}{x^6+1} - \frac{2x}{x^4+1}$$

Examples for part II

$$\int_a^b G'(x) dx = G(b) - G(a) = G(x) \Big|_a^b$$

$$\begin{aligned} \text{Eq 8: } \int_0^\pi \cos x dx &= \sin x \Big|_0^\pi \\ &= \sin \pi - \sin 0 = 0 \end{aligned}$$

$$\text{Eg 9 } \int_1^4 \left(\frac{3}{2} \sqrt{x} - \frac{4}{x^2} \right) dx$$

$$= \int_1^4 \left(\frac{3}{2} x^{\frac{1}{2}} - 4x^{-2} \right) dx$$

$$= \left(x^{\frac{3}{2}} + 4x^{-1} \right) \Big|_1^4$$

$$= 9 - 5 = 4$$

$$\text{Eg 10 } \int_0^1 \frac{1}{x^2+1} dx$$

$$= \tan^{-1} x \Big|_0^1 = \frac{\pi}{4} - 0$$

$$\text{Eg 11 } \int_{-1}^1 \sqrt{1-x^2} dx =$$
 

$$= \frac{\pi}{2}$$