# Antiderivative

<u>Def</u>: $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$

<u>Rm</u>: If $F(x)$ is an antiderivative of $f(x)$, then so is $F(x)+C$ for any constant $C$.

<u>Def</u> $\int f(x)\,dx$ = the collection of all $F(x)+C$, such that $F'(x)=f(x)$

i.e. = the collection of all antiderivatives of $f(x)$

Eg1 $f(x)$ $\int f(x)\,dx$

$$x^n \qquad \begin{cases} \frac{x^{n+1}}{n+1}+C, & n \neq -1 \\ \ln|x|+C, & n=-1 \end{cases}$$

$$\sin kx \qquad \frac{-1}{k}\cos kx + C$$

$$\cos kx \qquad \frac{1}{k}\sin kx + C$$

$$\sec^2 kx \qquad \frac{1}{k}\tan kx + C$$

---- etc.

$$e^{kx} \qquad \frac{1}{k}e^{kx}+C$$

$$a^{kx} \qquad \frac{1}{k\ln a}a^{kx}+C$$

$$(=(e^{\ln a})^{kx}=e^{k\ln a\,x}$$

Eg2 Solve $\begin{cases} \frac{dv}{dt} = -32 \\ v(0) = 12 \end{cases}$

$v(t) = ?$

<u>Sol</u>

$$v(t) = -32t + C$$

$$v(0) = 12$$

$$\Rightarrow C = 12$$

$$\Rightarrow v(t) = -32t + 12$$

Eg 3 Solve $\begin{cases} \dfrac{d^2x}{dt^2} = -9.8 \\ x(0) = 10 \\ x'(0) = 0 \end{cases}$

Sol

$$\frac{d^2x}{dt^2} = -9.8 \Rightarrow \frac{dx}{dt} = -9.8t + C_1$$

$$\Rightarrow x(t) = -4.9t^2 + C_1 t + C_2$$

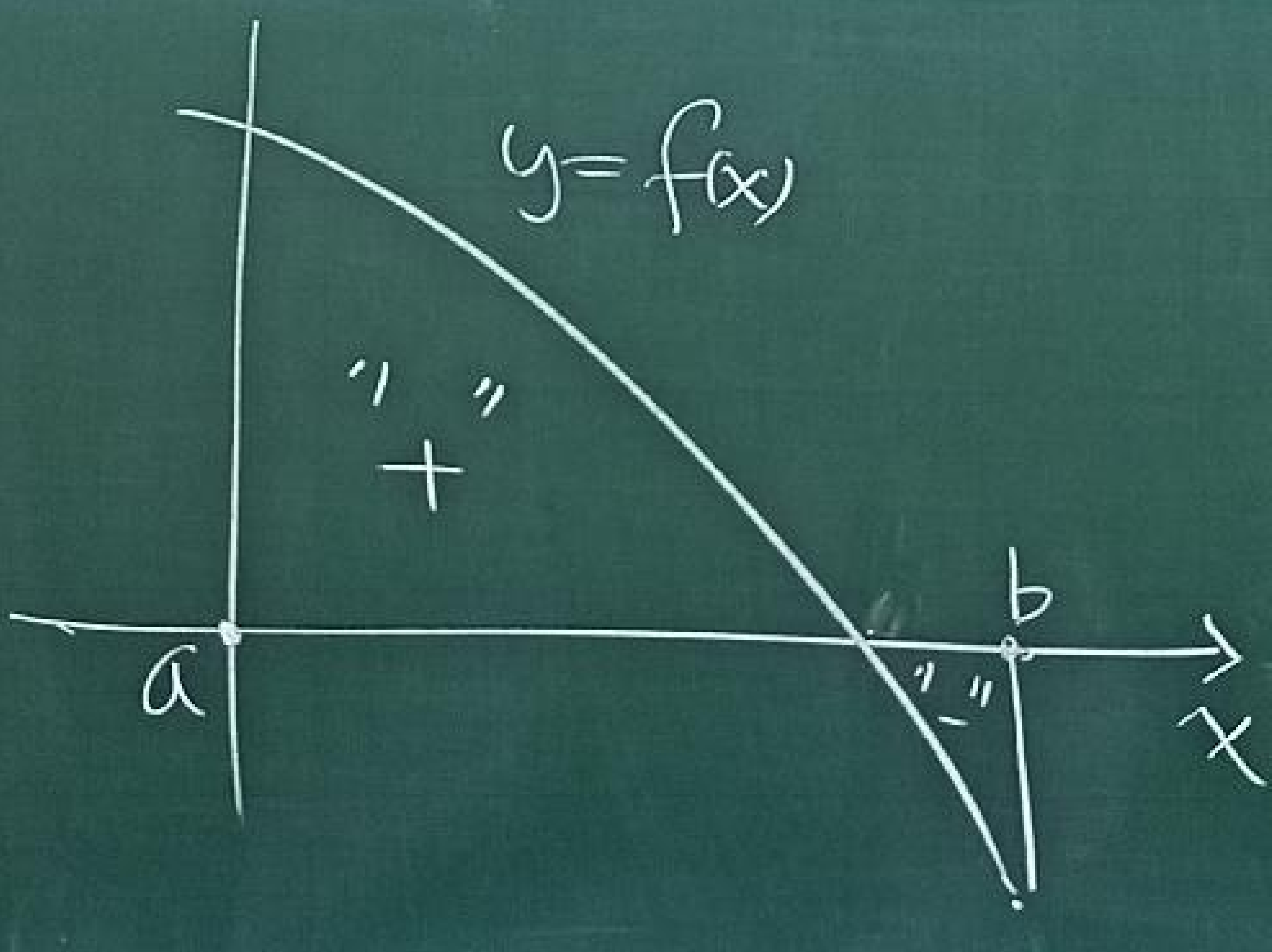
$$\begin{cases} x(0) = 10 \\ x'(0) = 0 \end{cases} \Rightarrow \begin{cases} C_2 = 10 \\ C_1 = 0 \end{cases}$$

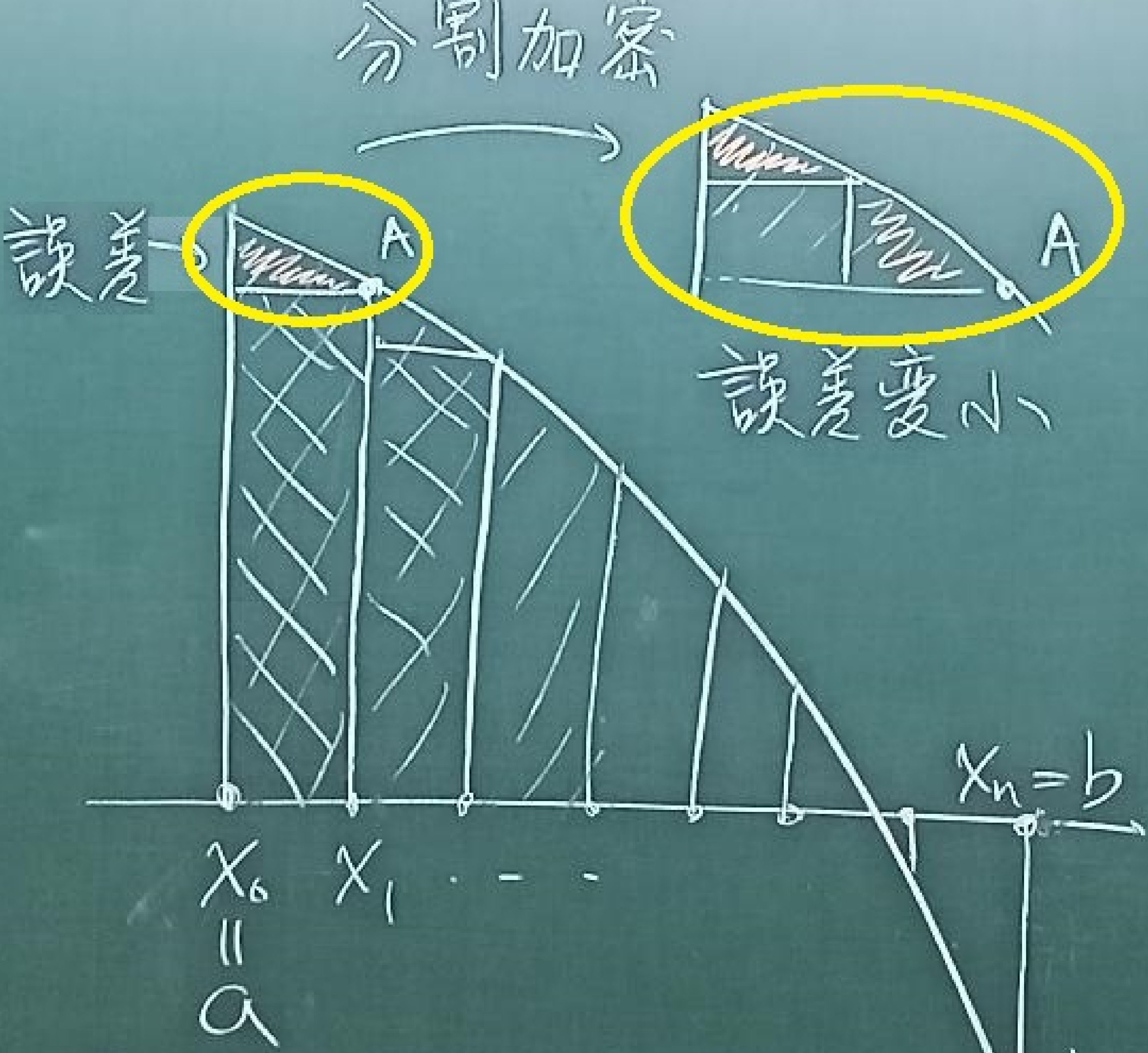
$$\therefore x(t) = -4.9t^2 + 10$$

Exercise: What if $\begin{cases} x(0) = 10 \\ x(1) = 0 \end{cases}$ ?

# Integration

How to compute the "signed area" under the graph of $f(x)$ on $[a,b]$






$$\mathbb{P}=\{a=x_0,\ x_1,\ x_2,\cdots x_n=b\}$$

$$a=x_0<x_1<x_2\cdots<x_n=b$$

For example, let $x_{i+1}=x_i+\dfrac{b-a}{n}$

(uniform partition)

We approximate the signed area by

$$A_n = \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k)$$

where $x_{k-1} \le c_k \le x_k$

Here in the picture,

$c_k = x_k$.

If $x_k = x_{k-1} + \frac{b-a}{n}$, $c_k = x_k$

We formally obtained the ~~sign~~ signed area by $\lim_{n\to\infty} A_n$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} (x_k - x_{k-1}) f(x_k)$$

# Definite Integral

$a = x_0 < x_1 \cdots < x_n = b$

$\mathbb{P} = \{x_0, x_1 \ldots x_n\}$ (Partition)

$\|\mathbb{P}\| \stackrel{\text{def}}{=} \max\limits_{1 \le k \le n} (x_k - x_{k-1})$

= length of the largest interval

Signed area is defined as

$$\int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{\|\mathbb{P}\| \to 0} \underbrace{\sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k)}_{\text{Riemann Sum}}$$

(where $c_k \in [x_{k-1}, x_k]$ is arbitrary)

provided the limit exist and we say f is integrable on [a, b]

Precise definition of

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k) = I$$

For any $\varepsilon > 0$, there exists a corresponding $\delta > 0$, such that

$$\|P\| < \delta \Rightarrow \left| \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k) - I \right| < \varepsilon$$

<u>Thm</u>: If $f$ is cont. on $[a,b]$ (or only a few jump discontinuities) then this limit exist.

**Rm** Why use $\lim\limits_{\|P\|\to 0}$, not $\lim\limits_{n\to\infty}$ as definition?

**Ans**: There are some strange functions where $\lim\limits_{n\to\infty}$ exists, but $\lim\limits_{\|P\|\to 0}$ does not exist. We want to exclude these functions, and ~~not call them integrable~~ not consider them.

Eg1 Let $f(x)=\begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$

$\mathbb{P}=\{a=x_0<x_1<\dots<x_n=b\}$

$x_k = a + k\frac{b-a}{n}$ (uniform partition)

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}f(c_k) \cdots (*)$$

Case (a). $c_k = x_k$ = rational $f(x_k)=1$, $(*)=1$

Case (b) $c_k = x_{k-1} + \frac{x_k - x_{k-1}}{\pi} = \frac{k-1}{n} + \frac{1}{n\pi}$ = irrational

$\cdot\ (*) = 0$

$(*)$ has different limit for different choice of $c_k$

$\Rightarrow \lim\limits_{n\to\infty}$ does not converge

$f(x)$ is not integrable on $[0,1]$

This is why $\lim_{n\to\infty}$ is not a good definition for $\int_a^b f(x)dx$

By requiring the limit exist for arbitrary $\mathbb{P}$ and $c_k$. we can exclude these strange functions from consideration

<u>Rm</u> $f(x)$ is integrable on $[a,b]$

If $\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k)$

exists

Recall

$$(*)\ \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta x_k f(c_k) = \int_a^b f(x)\,dx$$

where $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$

$$\Delta x_k = x_k - x_{k-1}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} f(x_k)$$

(if the limit $(*)$ exists)

= Signed area under graph of $y = f(x)$ between $a$ and $b$

**Eg2** $\int_0^b x^2\,dx = ?$

**Sol**: $x_k = \frac{bk}{n}$

$\Delta x_k = \frac{b}{n}$

$$\text{Ans} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b}{n}\left(\frac{bk}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{b^3}{n^3} \sum_{k=1}^{n} k^2$$

$$= \lim_{n\to\infty} \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{b^3}{3}$$

How about $\int_a^b x^2 dx$?

<u>Rm</u> $a<b$, $\int_b^a f(x)dx = ?$

We define $\int_b^a f(x)dx = -\int_a^b f(x)dx$

With this definition,
we always have

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

regardless of ordering of a, b, c.

i.e. $a<b<c$, $c<a<b$
$b<c<a$, $a<c<b$, etc.

**TABLE 5.6** Rules satisfied by definite integrals

| | | |
|---|---|---|
| 1. *Order of Integration:* | $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ | A definition |
| 2. *Zero Width Interval:* | $\int_a^a f(x)\,dx = 0$ | A definition when $f(a)$ exists |
| 3. *Constant Multiple:* | $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ | Any constant $k$ |
| 4. *Sum and Difference:* | $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ | |
| 5. *Additivity:* | $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ | |
| 6. *Max-Min Inequality:* | If $f$ has maximum value max $f$ and minimum value min $f$ on $[a, b]$, then $\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$ | |
| 7. *Domination:* | $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$ | |
| | $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ | (Special case) |

Eg 3. $\int_a^b x^2\,dx = ?$

<u>Sol</u>: $= \int_0^b x^2\,dx - \int_0^a x^2\,dx$

$= \frac{b^3}{3} - \frac{a^3}{3}$

Similar limiting process can be used to evaluate

$\int_0^1 x^n\,dx,\ n = 0, 1, 2, \cdots$

$\int_0^1 \begin{matrix}\sin x\\ \cos x\end{matrix}\,dx$, (need trigonometric identities ...)

$\int_0^1 e^x\,dx$, (sum of geometric series)

Eg4 $\int_0^b x^m dx = ?$ $m \in \mathbb{N}$

Sol $x^m$ is continuous $\therefore$ integrable

$\therefore \int_0^b x^m dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} (x_k - x_{k-1}) \cdot f(c_k)$

Can be computed by taking

$P = \{0 = x_0 < x_1 < \cdots < x_n = b\}$

with uniform partition ($x_k = \frac{kb}{n}$)

and convenient $c_k$ (take $c_k = x_k$)

$\therefore = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{b}{n} \left(\frac{kb}{n}\right)^m = \lim_{n \to \infty} \frac{b^m \sum_{k=1}^{n} k^m}{n^{m+1}}$

Trick: $k^{m+1} - (k-1)^{m+1} = (m+1)k^m - \frac{(m+1)m}{2} k^{m-1} + \cdots$

$\sum_{k=1}^{n} \Rightarrow n^{m+1} - 0 = (m+1) \sum_{k=1}^{n} k^m - \frac{(m+1)m}{2} \sum_{k=1}^{n} k^{m-1} + \cdots$

$\lim_{n \to \infty} \frac{1}{n^{m+1}} \times \Rightarrow 1 = (m+1) \lim \frac{\sum_{k=1}^{n} k^m}{n^{m+1}} + 0 + 0 + \cdots$

$\therefore \lim_{n \to \infty} \frac{\sum_{k=1}^{n} k^m}{n^{m+1}} = \frac{1}{m+1}$, $\therefore \int_0^b x^m dx = \frac{b^{m+1}}{m+1}$

Eg5 $\int_0^b \sin x\,dx = ?$

Sol: take $x_k = \frac{kb}{n}$, $\Delta x = x_k - x_{k-1} = \frac{b}{n}$

take $c_k = x_{k-1} + \frac{\Delta x}{2}$ $\left(= \frac{x_{k-1}+x_k}{2}\right)$

$$\int_0^b \sin x\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b}{n} \sin(\underbrace{x_{k-1} + \frac{\Delta x}{2}}_{= c_k})$$

$$\cos(x_k) - \cos(x_{k-1})$$

$$= \cos\left(c_k + \frac{\Delta x}{2}\right) - \cos\left(c_k - \frac{\Delta x}{2}\right)$$

$$= -2\sin c_k \cdot \sin\frac{\Delta x}{2}$$

$$\therefore \int_0^b \sin x\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b}{n} \cdot \frac{\cos x_k - \cos x_{k-1}}{-2\sin\left(\frac{\Delta x}{2}\right)}$$

$$= \lim_{n\to\infty} \frac{\frac{b}{n}}{-2\sin\left(\frac{b}{2n}\right)} \sum_{k=1}^{n} (\cos x_k - \cos x_{k-1})$$

$$= \lim_{x\to 0} \frac{x}{-2\sin\frac{x}{2}} (\cos x_n - \cos x_0) = -(\cos b - \cos 0)$$

Eg 6: $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{1-\left(\frac{k}{n}\right)^2}$

Sol write $\frac{1}{n} = x_k - x_{k-1}$

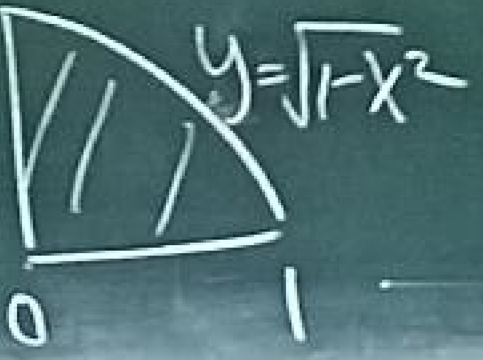
i.e. $x_k = \frac{k}{n}$

$\Rightarrow \mathbb{P} = \{0 = x_0 < x_1 \cdots < x_n = 1\}$

i.e. $[a, b] = [0, 1]$

$\sqrt{1-\left(\frac{k}{n}\right)^2} = \sqrt{1-x_k^2} = f(x_k)$

$\text{Ans} = \int_0^1 f(x)\,dx = \int_0^1 \sqrt{1-x^2}\,dx$

$=$ 

$= \frac{\pi r^2}{4} = \frac{\pi}{4}$