Eg10: $\lim\limits_{x\to 0} \dfrac{x^2\cos\frac{1}{x}}{\sin x} = ?$ $\left(\text{"}\dfrac{0}{0}\text{"}\right)$

$\overset{?}{=} \lim\limits_{x\to 0} \dfrac{2x\cos\frac{1}{x} + x^2\left(\frac{-1}{x^2}\right)\sin\frac{1}{x}}{\cos x}$

$= \lim\limits_{x\to 0} \dfrac{2x\cos\frac{1}{x} - \sin\left(\frac{1}{x}\right)}{\cos x}$ does not exist

$\overset{?}{\Rightarrow} \lim\limits_{x\to 0} \dfrac{x^2\cos\frac{1}{x}}{\sin x}$ does not exist

Ans: l'Hôpital Rule only applies to $\lim \dfrac{f'(x)}{g'(x)} = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$. It does not apply in this case.

$\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ does not exist $\not\Rightarrow$ $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ does not exist

Instead, we write

$$\lim_{x \to 0} \frac{x^2 \cos\frac{1}{x}}{\sin x}$$

$$= \lim_{x \to 0} \left(\frac{x}{\sin x}\right)\left(x \cos\frac{1}{x}\right)$$

$$= \left(\lim_{x \to 0} \frac{x}{\sin x}\right)\left(\lim_{x \to 0} x \cos\frac{1}{x}\right)$$

$$= 1 \cdot 0 = 0$$

Other applications

"1^{∞}", "0^{0}", "∞^{0}"

Eg11 $\lim_{x \to 0} (1+ax)^{\frac{1}{x}}$ ("1^{∞}")

$= \lim_{x \to 0} \left(e^{\ln(1+ax)}\right)^{\frac{1}{x}} = e^{\lim_{x \to 0} \frac{\ln(1+ax)}{x}}$

$= e^{\left(\lim_{x \to 0} \frac{a}{1+ax}\right)} = e^{a}$

Eg 12 $\lim_{x\to\infty} x^{\frac{1}{x}}$ "∞^0"

$$= \lim_{x\to\infty} \left(e^{\ln x}\right)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln x}{x}}$$

$$= e^{\lim_{x\to\infty} \frac{\ln x}{x}} \quad \left(e^{\text{"}\frac{\infty}{\infty}\text{"}}\right)$$

$$= e^{\lim_{x\to\infty} \frac{\frac{1}{x}}{1}} = 1$$

<u>Rm</u> For any $a, b, c > 0$

$$(|\ln x|)^a \ll x^{-b} \ll e^{\frac{c}{x}}$$

as $x \to 0^+$

Eg 13 $\lim\limits_{x\to 0^+} x^x$ "0^0"

Sol: $= \lim\limits_{x\to 0^+} \left(e^{\ln x}\right)^x$

$= \lim\limits_{x\to 0^+} e^{x\ln x}$ "$e^{-0\cdot\infty}$"

$= \lim\limits_{x\to 0^+} e^{\frac{\ln x}{\frac{1}{x}}}$ "$e^{\frac{-\infty}{\infty}}$"

$= e^{\lim\limits_{x\to 0^+}\left(\frac{\ln x}{x^{-1}}\right)}$

$= e^{\lim\limits_{x\to 0^+} \frac{x^{-1}}{-x^{-2}}} = e^{\lim\limits_{x\to 0^+}(-x)} = 1$

Proof of l'Hôpital's Rule

(the $\lim_{x\to a}$ "$\frac{0}{0}$" version)

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

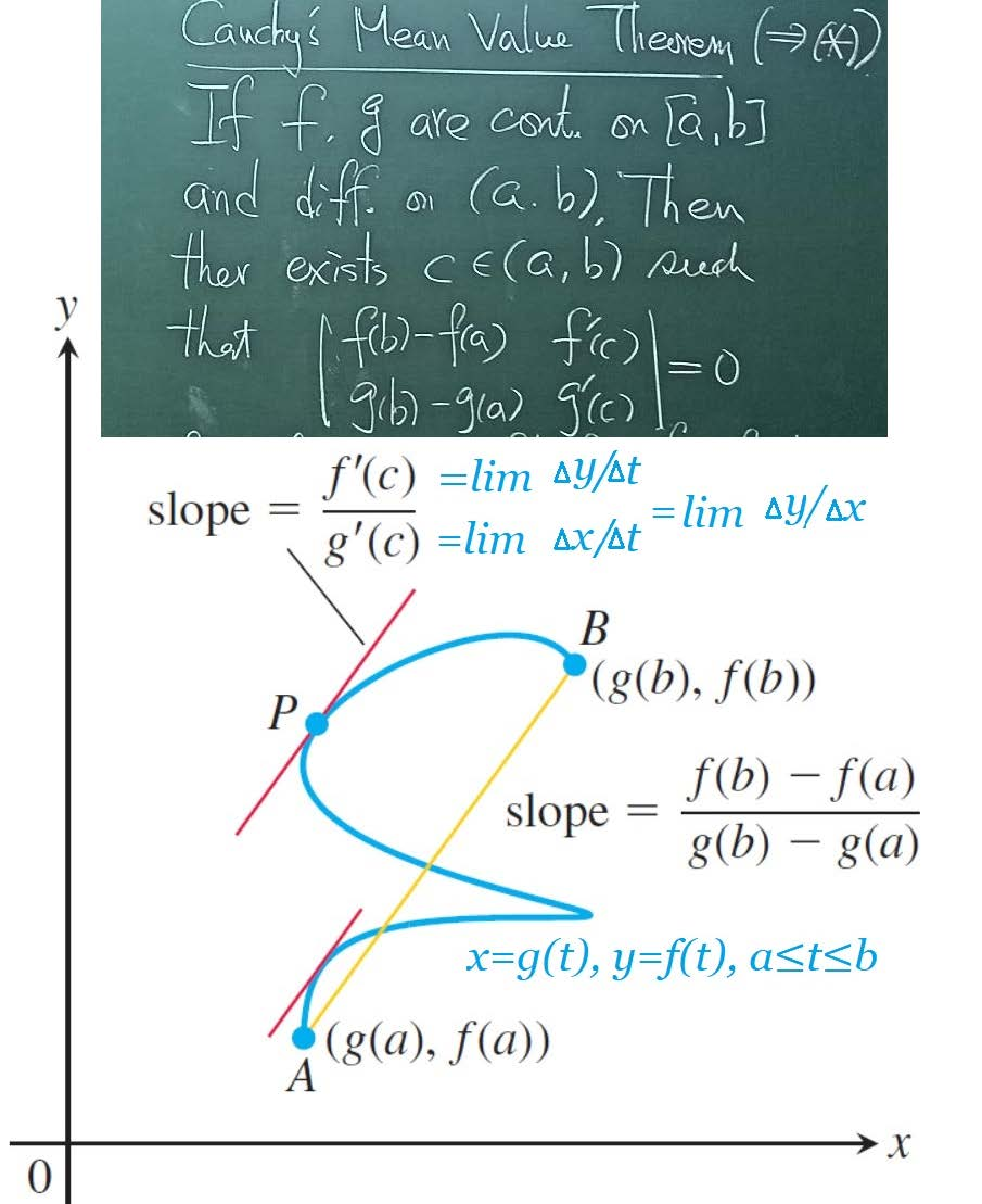
MVT?

$$\underset{(*)}{\overset{??}{=}} \lim_{x\to a}\frac{f'(c)}{g'(c)}, \text{ for some } c \text{ between } a, x$$

$$\underset{(**)}{=} \lim_{x\to a}\frac{f'(x)}{g'(x)} \left(\text{if this limit } = \begin{cases} L \\ +\infty \\ -\infty \end{cases}\right)$$

No!

MVT $\not\Rightarrow$ (*) Since the "c" may be different for f and g



Cauchy's Mean Value Theorem (⇒(*))
If f, g are cont. on [a,b]
and diff. on (a.b). Then
ther exists c ∈ (a,b) such
that $\begin{vmatrix} f(b)-f(a) & f'(c) \\ g(b)-g(a) & g'(c) \end{vmatrix} = 0$
y
slope = $\frac{f'(c)}{g'(c)}$ =lim Δy/Δt
=lim Δx/Δt
=lim Δy/Δx
B
(g(b), f(b))
P
slope = $\frac{f(b)-f(a)}{g(b)-g(a)}$
x=g(t), y=f(t), a≤t≤b
(g(a), f(a))
A
x
0

pf: Let $G(x) = \begin{vmatrix} f(b)-f(a) & f(x)-f(a) \\ g(b)-g(a) & g(x)-g(a) \end{vmatrix}$

$\Rightarrow G(a)=0,\ G(b)=0$

$\xRightarrow{\text{Rolle's}} \exists c \in (a,b),\ 0 = G'(c) = \begin{vmatrix} f(b)-f(a) & f'(c) \\ g(b)-g(a) & g'(c) \end{vmatrix}$

Rm Textbook version (assuming $g'(x) \neq 0$ near $x=a$)

Let $F(x) = f(x) - f(a) - \dfrac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$

$\Rightarrow F(a) = 0,\ F(b) = 0$.

$\xRightarrow{\text{Rolle's}} \exists c \in (a,b),\ F'(c) = 0$

i.e. $f'(c) - \dfrac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0$

$\Rightarrow \dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

pf of l'Hôpital's Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} \overset{*}{=} \lim_{x\to a}\frac{f'(c)}{g'(c)}$$

$$\left(\begin{array}{l} \text{CMVT} \\ \Longrightarrow \end{array} \begin{vmatrix} f(b)-f(a) & f'(c) \\ g(b)-g(a) & g'(c) \end{vmatrix} = 0 \quad c\in(a,b) \right.$$

$$\left. \therefore \begin{vmatrix} f(x)-f(a) & f'(c) \\ g(x)-g(a) & g'(c) \end{vmatrix} = 0, \ \begin{array}{l} c\in(a,x) \\ \text{or } c\in(x,a) \end{array} \Rightarrow * \right)$$

$$\therefore \lim_{\underline{x\to a}}\frac{f'(x)}{g'(x)} = \begin{cases} L \\ +\infty \\ -\infty \end{cases} \Rightarrow \lim_{\underline{x\to a}}\frac{f'(c)}{g'(c)} = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$$

$$\Longrightarrow \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$$

<u>Rm</u> From the proof of l'Hôpital's Rule, it is clear that if

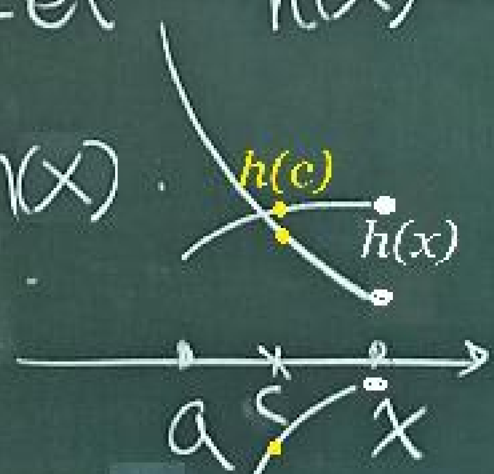
$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$$

then

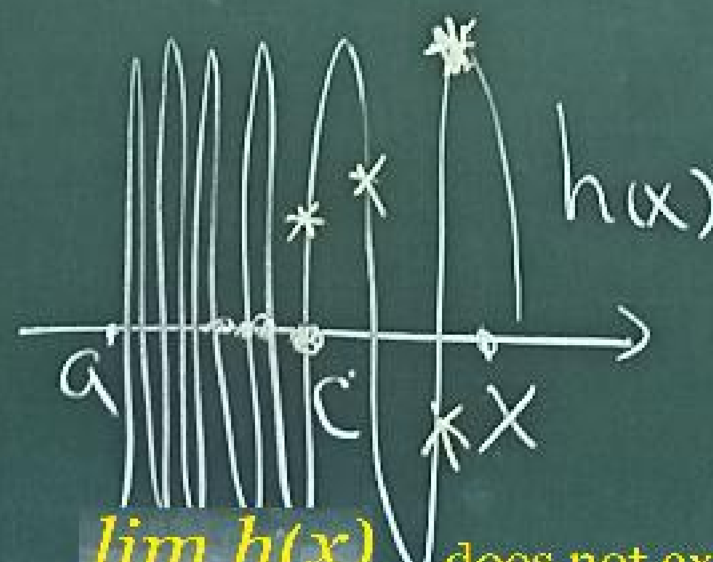
$$\lim_{x\to a} \frac{f'(c)}{g'(c)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad (**)$$

for any c between a and x

Let $h(x) = \dfrac{f'(x)}{g'(x)}$

$h(x)$: 

$\begin{cases} L \\ +\infty \\ -\infty \end{cases}$



$\lim\limits_{x\to a} h(x) = L, \pm\infty$

$\Longrightarrow \lim\limits_{x\to a} h(c) = L, \pm\infty$

$\lim\limits_{x\to a} h(x)$ does not exist

$\not\Longrightarrow \lim\limits_{x\to a} h(c)$ does not exist

However, if $\lim_{x \to a} h(x)$ is oscillatory (or does not exist)

and c is any point between a and x

It is possible that

$\lim_{x \to a} \frac{f'(c)}{g'(c)}$ converges.

$\underline{\text{Eg}}$ $h(x) = \sin\left(\frac{1}{x}\right)$

c between 0 and x

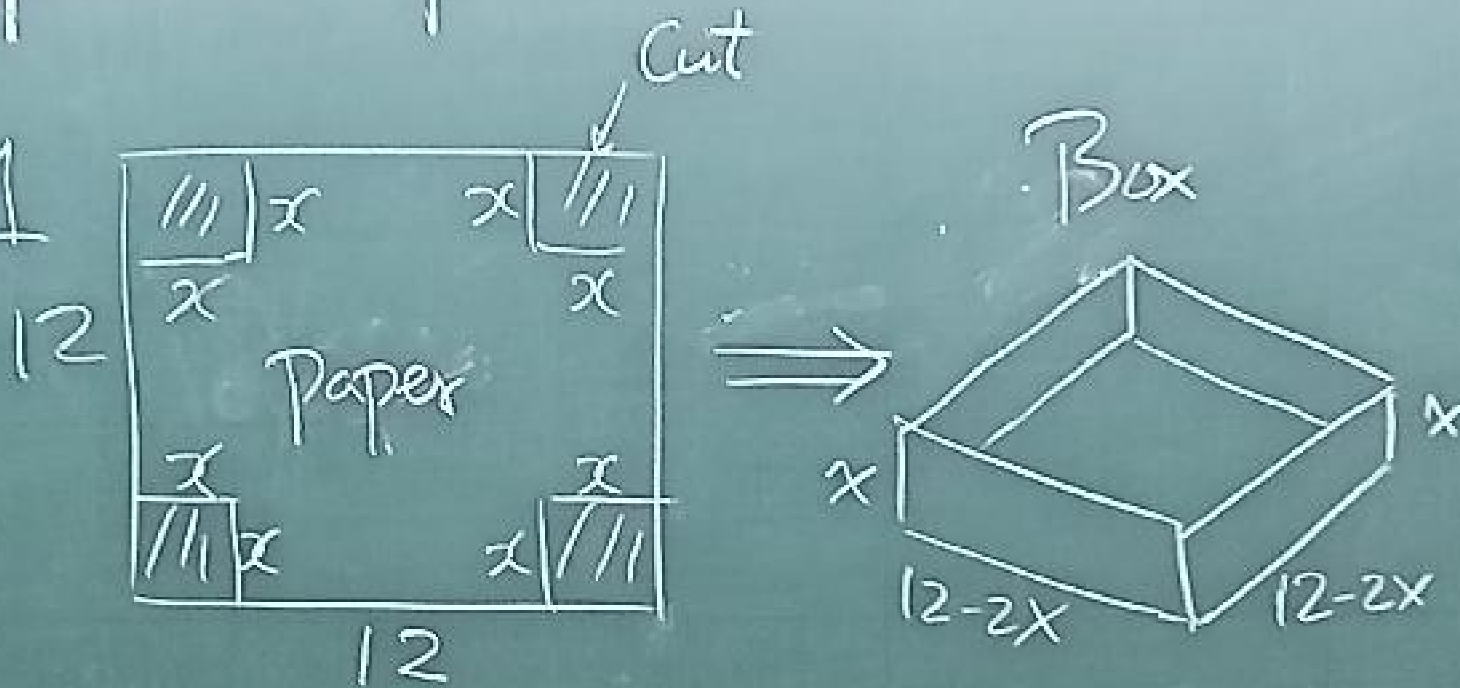
with $\sin\left(\frac{1}{c}\right) = 0$

i.e.

lim f'(x)/g'(x) DNE $\not\Rightarrow$ lim f(x)/g(x) DNE

Applied Optimization

Eg1



Find maximal volume of the box.

Sol: $V(x) = (12-2x)^2 \cdot x$

find global max on $V(x)$ on $0 \le x \le 6$

$V'(x) = 12x^2 - 96x + 144 = 12(x-2)(x-6)$

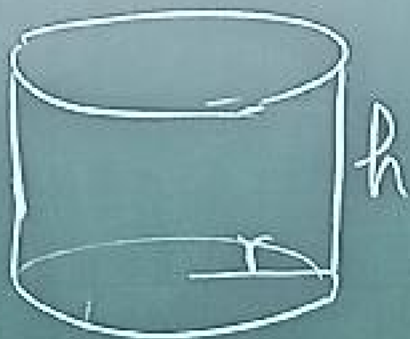
$V'(x)$	+		$-$		+	
	0	2		6		
$V(x)$	$\nearrow$	2	$\searrow$	6	$\nearrow$	

local min: $f(0), f(6)$, local max: $f(2)$

Compare values of f on critical point $(x=2)$ and boundary $(x=0,6)$

$\Rightarrow$ global max $= f(2)$

Eg2.



Fix $V = \pi r^2 h = 1000$
find minimal surface area of the cylinder.

Sol. $A = 2\cdot\pi r^2 + 2\pi r \cdot h$

$$= 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2}$$

$\therefore A(r) = 2\left(\pi r^2 + \frac{1000}{r}\right)$ on $r > 0$

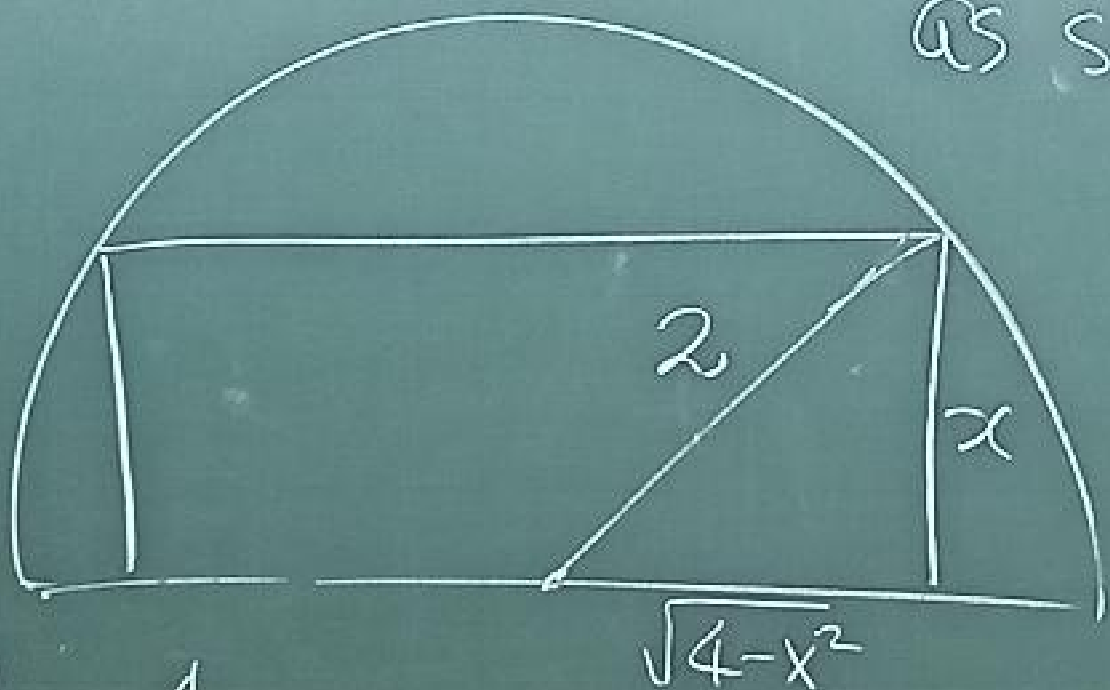
$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4\pi\left(r^3 - \frac{500}{\pi}\right)}{r^2}$$

r^2 always > 0, $r^3 > \frac{500}{\pi} \Leftrightarrow r > \left(\frac{500}{\pi}\right)^{\frac{1}{3}}$

$A'(r)$	$-$		$+$
	0	$\left(\frac{500}{\pi}\right)^{\frac{1}{3}}$	$\rightarrow$
$A(r)$	$\searrow$		$\nearrow$

global min $= A\left(\left(\frac{500}{\pi}\right)^{\frac{1}{3}}\right)$

Eg 3: Maximize the area of the rectangle as shown below.



<u>Ans</u>: global max $= A(\sqrt{2}) = 4$.

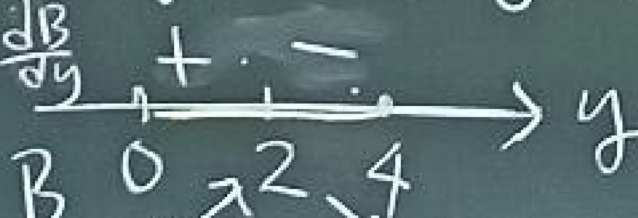
<u>Sol</u>: $A(x) = 2\sqrt{4-x^2}\cdot x \geq 0$

Maximize $A(x)$ on $0 \leq x \leq 2$

<u>Trick</u>: Maximize $B = A^2(x) = 4(4-x^2)x^2$ on $0 \leq x \leq 2$

$= $ Maximize $4(4-y)y$ on $0 \leq y \leq 4$

$\frac{dB}{dy} = 8\cdot(2-y)$



$\Rightarrow$ Global max $\Leftrightarrow y = 2 \Leftrightarrow x = \sqrt{2}$

Eg 4. x : # of product, $\underline{x \geq 0}$

$$r(x) = 9x \ : \ \text{revenue}$$

$$C(x) = x^3 - 6x^2 + 15x \ : \ \text{cost}$$

Find maximal profit

$$p(x) = r(x) - C(x) = -x^3 + 6x^2 - 6x$$
$$= -x(x^2 + 6x - 6)$$

<u>Sol</u>. $p'(x) = -3x^2 + 12x - 6$

$$= -3(x - x_-)(x - x_+)$$

where $x_{\pm} = 2 \pm \sqrt{2}$

$p'(x)$		$-$	$+$	$-$	
	0		x_-		x_+
$p(x)$		$\searrow$	$\nearrow$	$\searrow$	

local max

$p(0)$, $p(x_+)$

Check $p(0) = 0$

$p(x_+) \overset{?}{\geq} 0$

Which is abs. max? $p(0)$ or $p(x_+)$?

Is $p(x_+)$ positive?

case (a) or case (b) below?

Ans: $p(x) = -x(x^2+6x-6)$

$$x^2+6x-6 = (x+3)^2-15$$

has 2 real roots

$\therefore$ $p(x)$ has 3 real roots (i.e. case (a))

$\therefore$ $p(x_+)>0$ = global max