First derivative test for local extrema

Suppose f is cont. on $[a,b]$,
c is a critical point, and
f is differentiable on $(a,c)\cup(c,b)$

Then

(i) $f'(x)$ "+" | "−" $\Rightarrow$ local max
(ii) $f'(x)$ "−" | "+" $\Rightarrow$ local min

c

$f(x)$ (i) $\nearrow$ $\searrow$; (ii) $\searrow$ $\nearrow$

(iii) $f'(x)$ "+" | "+" ; "−" | "−" $\Rightarrow$ not a local extrema ; not a local extrema

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$f(x)$ $\nearrow$ $\nearrow$; $\searrow$ $\searrow$

Remark Second derivative test

Suppose f'' is cont. on $(c-\delta, c+\delta)$ $\delta > 0$

$\left(\begin{smallmatrix} \mathrm{I} \\ \mathrm{II} \end{smallmatrix}\right)$ If $f'(c)=0$, $f''(c) \begin{smallmatrix} > 0 \\ < 0 \end{smallmatrix}$

then f has a local $\begin{smallmatrix} \text{min} \\ \text{max} \end{smallmatrix}$ at c

(iii) If $f'(c)=0$, $f''(c)=0$

$\Longrightarrow$ No conclusion.

Eg 1. $f_1(x)=x^3$, $f_2(x)=-x^3$

$f_3(x)=x^4$, $f_4(x)=-x^4$

First Derivative Test $\Longrightarrow$ f_1, f_2: Not local extrema at $x=0$, f_3: $x=0$: local min, f_4: $x=0$: local max

Eg2. Find all critical points of $f(x) = x^{\frac{1}{3}}(x-4)$ and determine whether they are local $\begin{smallmatrix}\text{min}\\\text{max}\end{smallmatrix}$ or neither.

<u>Sol</u>: $f(x) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$

$$f'(x) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{\frac{-2}{3}} = \frac{4}{3}x^{\frac{-2}{3}}(x-1)$$

Critical points: 0, 1

		0		1	
$x^{\frac{-2}{3}}$	+		+		+
$(x-1)$	$-$		$-$		+
$f'(x)$	$-$		$-$		+
$f(x)$	$\searrow$	0	$\searrow$	1	$\nearrow$
		Neither		local min	

Concavity and curve sketching

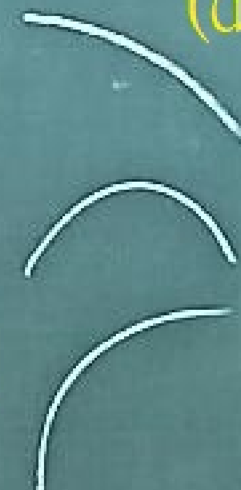
$f'' \begin{array}{l} >0 \\ <0 \end{array} \Rightarrow f'$ is $\begin{array}{l} \text{inc.} \\ \text{dec.} \end{array}$

concave up (upward)

concave down (downward)

concave up: upward

, concave down downward



<u>Def</u>. $(c, f(c))$ is called a point of inflection of f if f changes concavity across c

i.e. f is concave $\begin{array}{l} \text{upward} \\ \text{downward} \end{array}$ on $\begin{pmatrix} x<c \\ x>c \end{pmatrix}$ or $\begin{pmatrix} x>c \\ x<c \end{pmatrix}$ near $x=c$

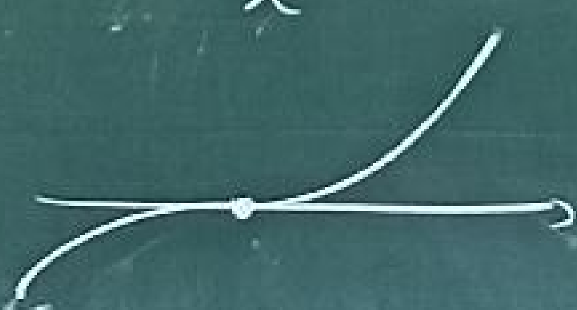
Eg 3. Is $(0, f(0))$ point of inflection?

$f(x) =$	$x^{\frac{5}{3}}$	x^4	x^3
	0	0	0
$f''(x)$	$-\ \times\ +$	$+\ 0\ +$	$-\ 0\ +$
point of inflection	Yes	No.	Yes

$$f''(c)=0 \ \overset{(x^4,\,x^3)}{\not\Longrightarrow}\ (c, f(c)) \text{ is a point of inflection}$$

$$f''(c)=0 \ \underset{(x^{\frac{5}{3}})}{\not\Longleftarrow}\ (c, f(c)) \text{ is a point of inflection}$$

$x^{\frac{5}{3}}$



Eg1 Sketch the graph of

$$f(x) = x^4 - 4x^3 + 10$$

and mark all local extrema and points of inflection.

Sol

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

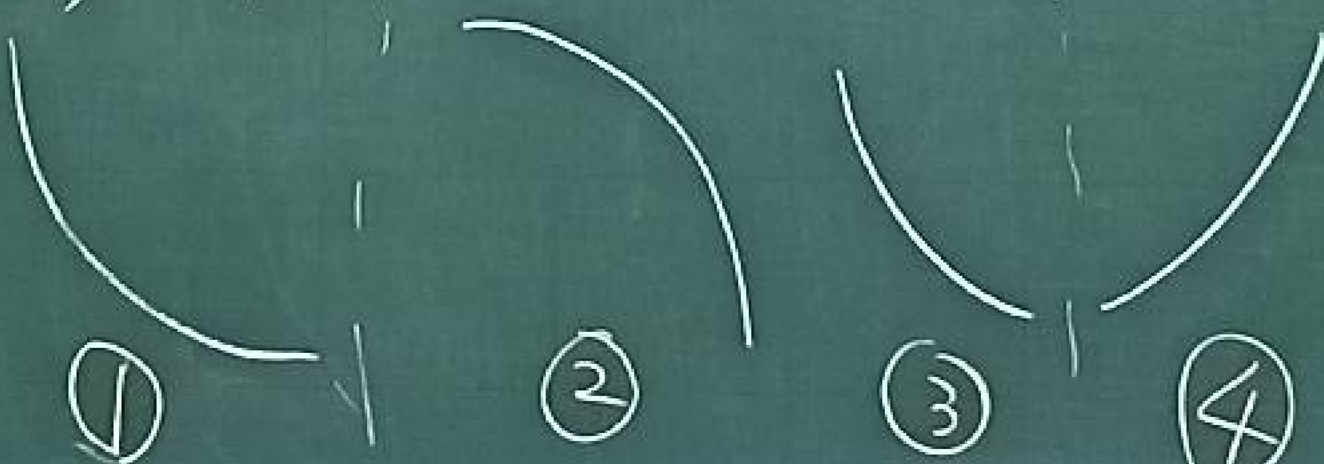
$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

$f'(x) = 0$. $x = 0, 3$ $\quad f(0) = 10$

$f''(x) = 0$, $x = 0, 2$ $\quad f(2) = -6$

$f(3) = -17$

x^2	+		+		+		+
$x-3$	−		−		−		+
$f'(x)$	−		−		−		+
		0		2		3	
x	−		+		+		+
$x-2$	−		−		+		+
$f''(x)$	+		−		+		+



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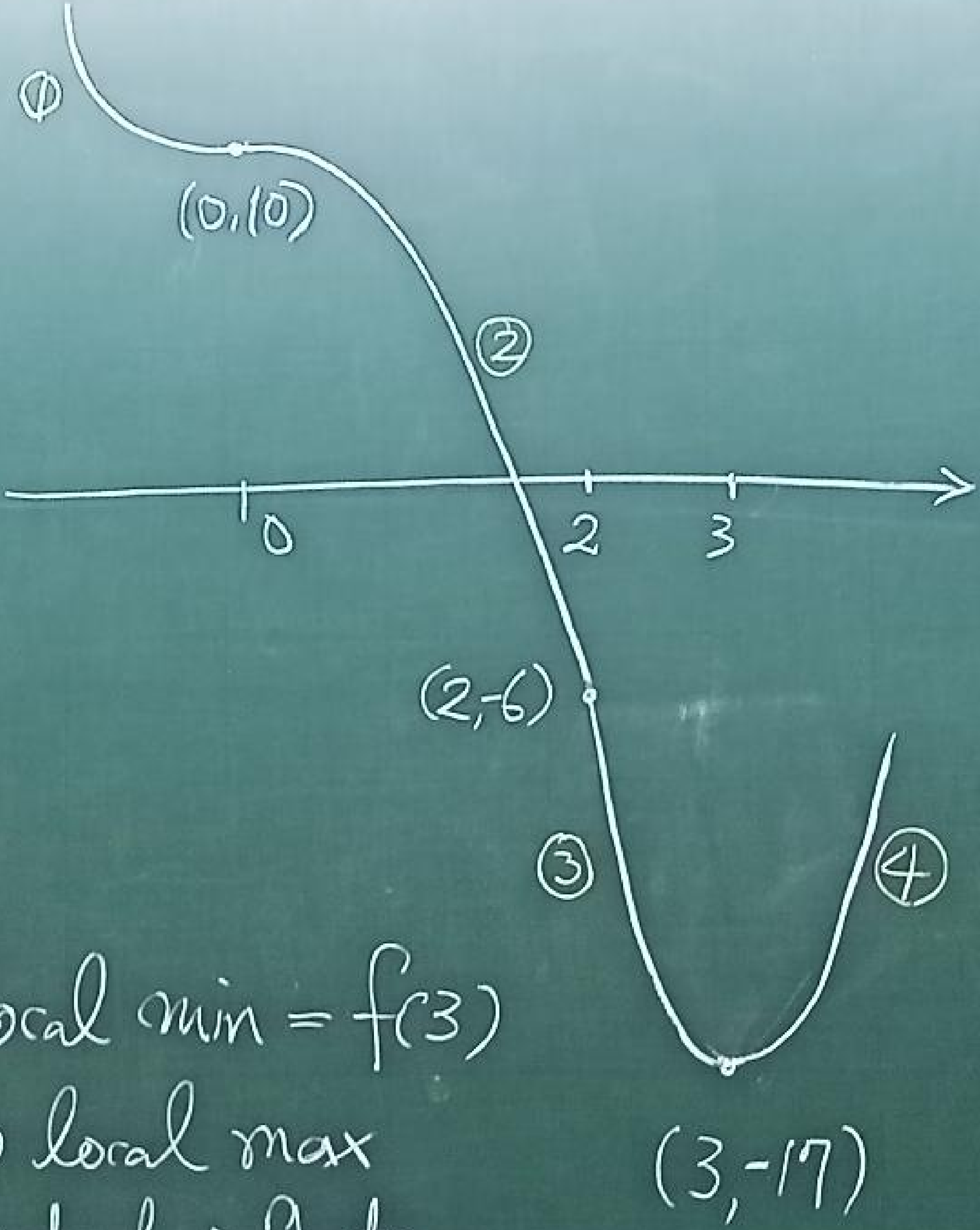


local min = $f(3)$

no local max

point of inflection

$(0, f(0)), (2, -6)$

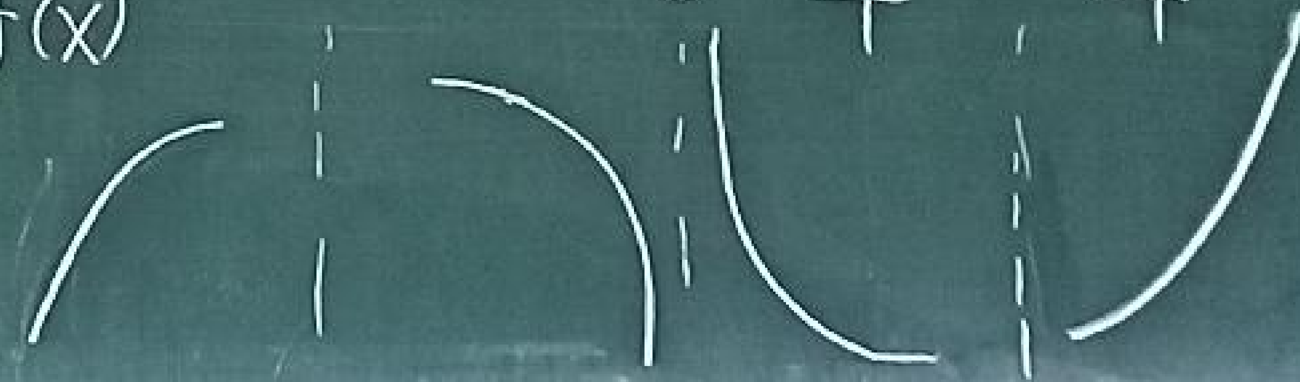
Eg 2 Sketch the graph of $f(x) = \dfrac{x^2+4}{2x}$

Sol $f(x) = \dfrac{x}{2} + \dfrac{2}{x}$

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{(x-2)(x+2)}{2x^2}$$

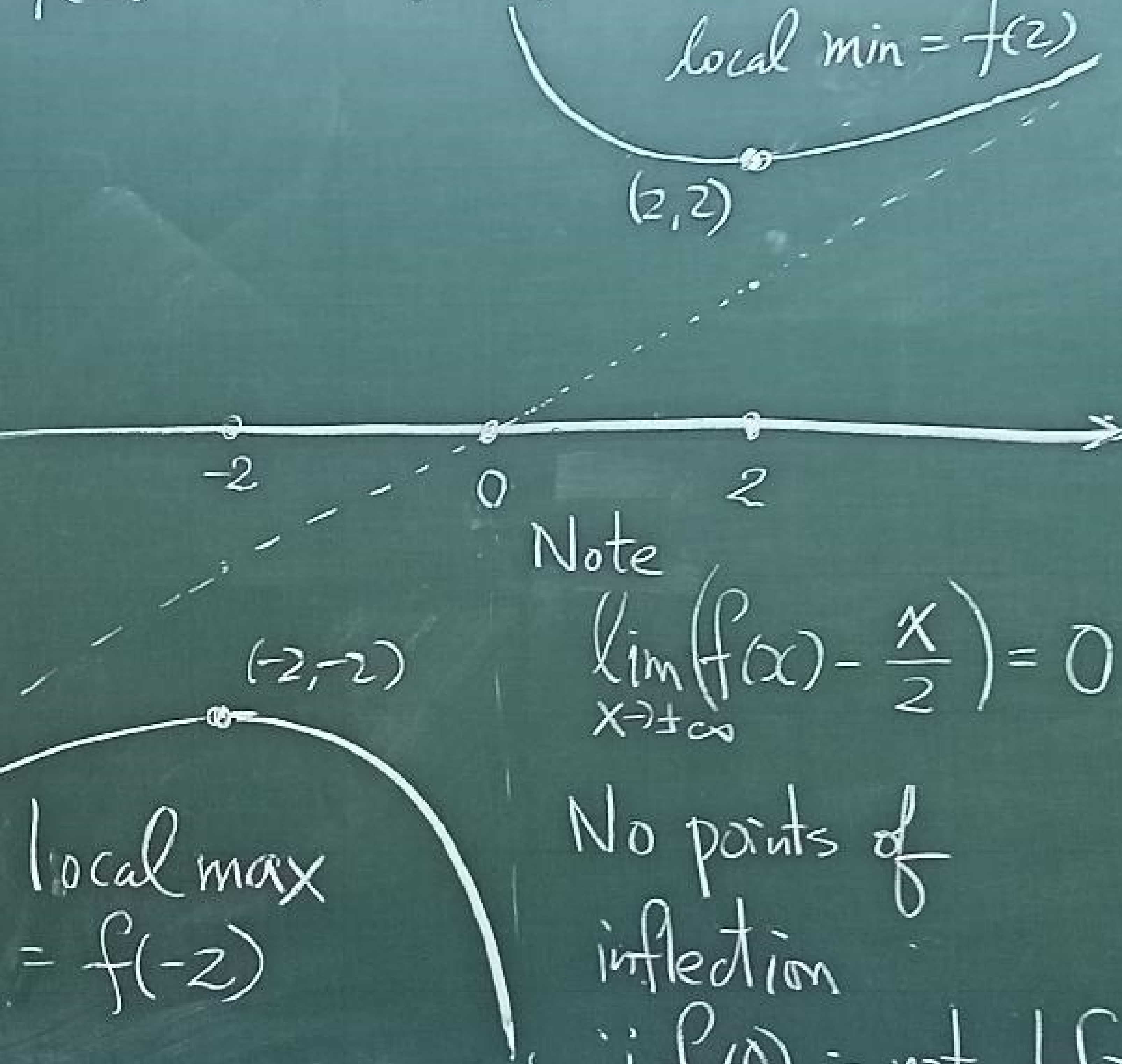
$$f''(x) = 4x^{-3}$$

x^2	+	+	+	+
$x-2$	−	−	−	+
$x+2$	−	+	+	+
$f'(x)$	+	−	−	+
	-2	0	2	
$f''(x)$	−	−	+	+



$f(0^{\pm}) = \pm\infty$,

$f(2) = 2$, $f(-2) = -2$



Note

$$\lim_{x \to \pm\infty} \left(f(x) - \frac{x}{2} \right) = 0$$

No points of inflection

$\because f(0)$ is not defined

Eg3 Sketch the graph of

$$f(x) = x^{\frac{1}{3}}(x-4)$$

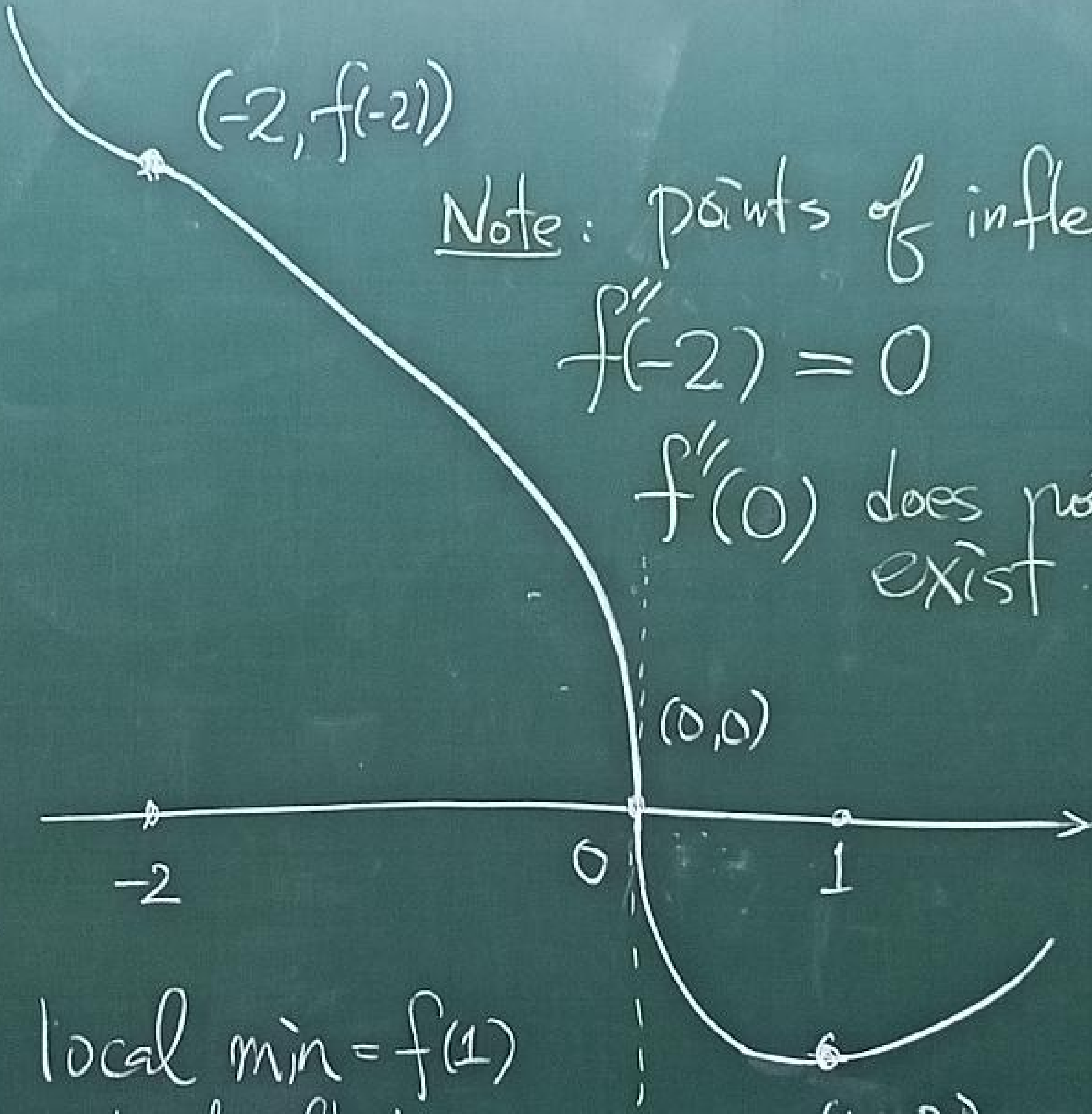
Sol $f(x) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$

$$f'(x) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{\frac{-2}{3}} = \frac{4}{3}x^{\frac{-2}{3}}(x-1)$$

$$f''(x) = \frac{4}{9}x^{\frac{-2}{3}} + \frac{8}{9}x^{\frac{-5}{3}} = \frac{4}{9}x^{\frac{-5}{3}}(x+2)$$

		-2		0		1	
$x^{\frac{2}{3}}$	+		+		+		+
$x-1$	−		−		−		+
$f'(x)$	−		−		−		+
$x^{\frac{5}{3}}$	−		−		+		+
$x+2$	−		+		+		+
$f''(x)$	+		−		+		+

$f(0)=0$, $f(1)=-3$, $f(-2)=6\sqrt[3]{2}$



Note: points of inflection

$f''(-2)=0$

$f''(0)$ does not exist

local min = $f(1)$

point of inflection

$(-2, f(-2))$, $(0, f(0))$