DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

When does f have abs. $\begin{matrix} \max \\ \min \end{matrix}$ on D?

Eg1. $f_1 : \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \longmapsto \mathbb{R}$

$$f_1(x) = \tan x$$

Eg2: $f_2 : (0, \infty) \longmapsto \mathbb{R}$

$$f_2(x) = \frac{1}{x}$$

Both f_1 and f_2 are continuous. But Neither has abs. $\begin{matrix} \max \\ \min \end{matrix}$

[The Extreme Value Theorem]

Thm1: (pf beyond this course)

If f is cont. on $[a,b]$

Then there exist

$x_m, x_M \in [a,b]$, such that

$f(x_m) = m$, $f(x_M) = M$

are abs. min and abs. max

i.e.

$m = f(x_m) \leq f(x) \leq f(x_M) = M$

for all $x \in [a,b]$

Thm 2. If

(i) $f : [a,b] \mapsto \mathbb{R}$ is cont.

(ii) f has a local $\begin{matrix} \text{min} \\ \text{max} \end{matrix}$ at $c \in (a,b)$

(iii) f is differentiable at c

$$\Longrightarrow f'(c) = 0$$
$$\not\Longleftarrow$$

Eg 3: $f : [-1,1] \mapsto \mathbb{R}$

$f(x) = x^3$, $f'(0) = 0$

but 0 is neither local $\begin{matrix} \text{min} \\ \text{max} \end{matrix}$

Pf of Thm 2

If f has a local min at c and $f'(c)$ exists, then

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0$$

$$f'(c) = \lim_{y \to c^-} \frac{f(y) - f(c)}{y - c} \leq 0$$

$$(y < c < x)$$

$$\Rightarrow f'(c) = 0.$$

Similarly for local max #

$f : [a,b] \longmapsto \mathbb{R}$ continuous

How do we find abs. $\begin{matrix} \text{max} \\ \text{min} \end{matrix}$?

<u>Note</u>: <u>Possible</u> abs. $\begin{matrix} \text{min} \\ \text{max} \end{matrix}$ include

critical points (背) $\left\{ \begin{array}{l} \text{(i) } c \in (a,b),\ f'(c) = 0 \\ \text{(ii) } c \in (a,b),\ f'(c) \text{ does not exist} \end{array} \right.$

(iii) $c = a$ or $c = b$

<u>Ans</u>: Step 1: find all candidates in (i), (ii) (iii)

<u>Step 2</u>: Compare values of f

Eg4 Find abs. $\begin{matrix} \text{min} \\ \text{max} \end{matrix}$

for $f(x) = x^{\frac{2}{3}}$ on $[-2, 3]$

<u>Sol</u>: Critical points ((ii) & (iii))

$f'(x) = \frac{2}{3} x^{\frac{-1}{3}}$

$= 0$ or does not exist

$\Rightarrow x = 0$

Possible candidates

$x = 0, \; -2, \; 3$

$f(x) = 0, \; \sqrt[3]{4}, \; \sqrt[3]{9}$

abs. min $\qquad$ abs. max

Rolle's Thm.

Suppose $h(x)$ is cont. on $[a,b]$
diff. on (a,b) and $h(a)=h(b)$
Then there exists $c\in(a,b)$
such that $h'(c)=0$

Mean Value Thm:

If $f(x)$ is cont. on $[a,b]$ and
diff. on (a,b), then there exists
$c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Obviously M.V.T. $\Rightarrow$ Rolle's

In fact, we also have " $\Leftarrow$ "

Pf: Suppose Rolle's Thm holds.

Let $h(x) = f(x) - g(x)$

where $g(x) = f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)$

$\Rightarrow g(a) = f(a),\ g(b) = f(b)$

$\Rightarrow h(a) = 0,\ h(b) = 0$

$\xrightarrow{\text{Rolle's}} \exists \in (a,b)$, such that

$0 = h'(c) = f'(c) - g'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a}$ #

Pf of Rolle's Thm.

Suppose $h\begin{pmatrix} x_m \\ x_M \end{pmatrix} = $ abs. $\begin{matrix} \text{min} \\ \text{max} \end{matrix}$

on $[a, b]$. x_m, x_M could be

(i) $c \in (a, b)$ $h'(c) = 0$

(ii) $c \in (a, b)$. $h'(c)$ does not exist

(iii) $c = a$ or b

(ii) is not possible since h is diff.

Case I: $x_m \in$ (i) or $x_M \in$ (i)

$\Rightarrow h'(x_m) = 0$ or $h'(x_M) = 0$.

$\Rightarrow c = x_m$ or x_M

Case II: $x_m = a, x_M = b$ or $x_m = b, x_M = a$

$h(a) = h(b) \Rightarrow h(x_m) = h(x_M)$

$\Rightarrow h(x) = \text{constant} \Rightarrow h'(c) = 0$ for all $c \in (a, b)$

Corollaries of M.V.T.

<u>Cor. 1</u> If $f'(x) \equiv 0$ on $(a.b)$

then $f(x) = \text{constant}$ on $(a.b)$

<u>Pf</u>: If $a < x_1 < x_2 < b$

and $f(x_1) \neq f(x_2)$.

$$\xrightarrow{\text{M.V.T}} \exists\, c \in (x_1, x_2) \text{ such that}$$

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0, \ \underline{\underline{\text{contradiction}}}.$$

<u>Cor. 2</u> If $f'(x) \equiv g'(x)$ on $(a.b)$,

then $f(x) = g(x) + C$ on $(a.b)$

Cor 3: Suppose f is cont. on $[a,b]$ and differentiable on (a,b).

If $f'(x) \gtrless 0$ for every $x \in (a,b)$ then $f(x)$ is $\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}$ on $[a,b]$

$\left(\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix} : \ x_1 < x_2 \Rightarrow f(x_1) \lesseqgtr f(x_2)\right)$

Pf: (for the "$>0 \Rightarrow$ increasing" case)

If not true (">0" and not increasing)

$\Rightarrow \exists x_1, x_2,\ a \leq x_1 < x_2 \leq b\ \ f(x_1) \geq f(x_2)$

$\xrightarrow{MVT} \exists c \in (x_1, x_2).\ f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \leq 0$. contradiction

Eg1 Show that

$f(x) = x^3 + 3x^2 + 1 = 0$ has exactly one root.

Sol Step 1: At least one root:

$f(0) = 1$, $f(-4) = -15$

$\xRightarrow{I.V.T} \exists\, x \in (-4, 0)$, $f(x) = 0$

Step 2 At most one root.

Suppose $x_1 < x_2$, $f(x_1) = f(x_2) = 0$

$\xRightarrow{MVT} \exists\, c \in (x_1, x_2)$, $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$

But $f'(c) = 3c^2 + 3 \geq 3 > 0$, Contradiction