Linearization (SKIP "differential")

Def: If $f(x)$ is differentiable at $x=a$, then the linear function

$$L(x) = f(a) + f'(a)(x-a)$$

is called the linearization of $f(x)$ centered at a (near a)

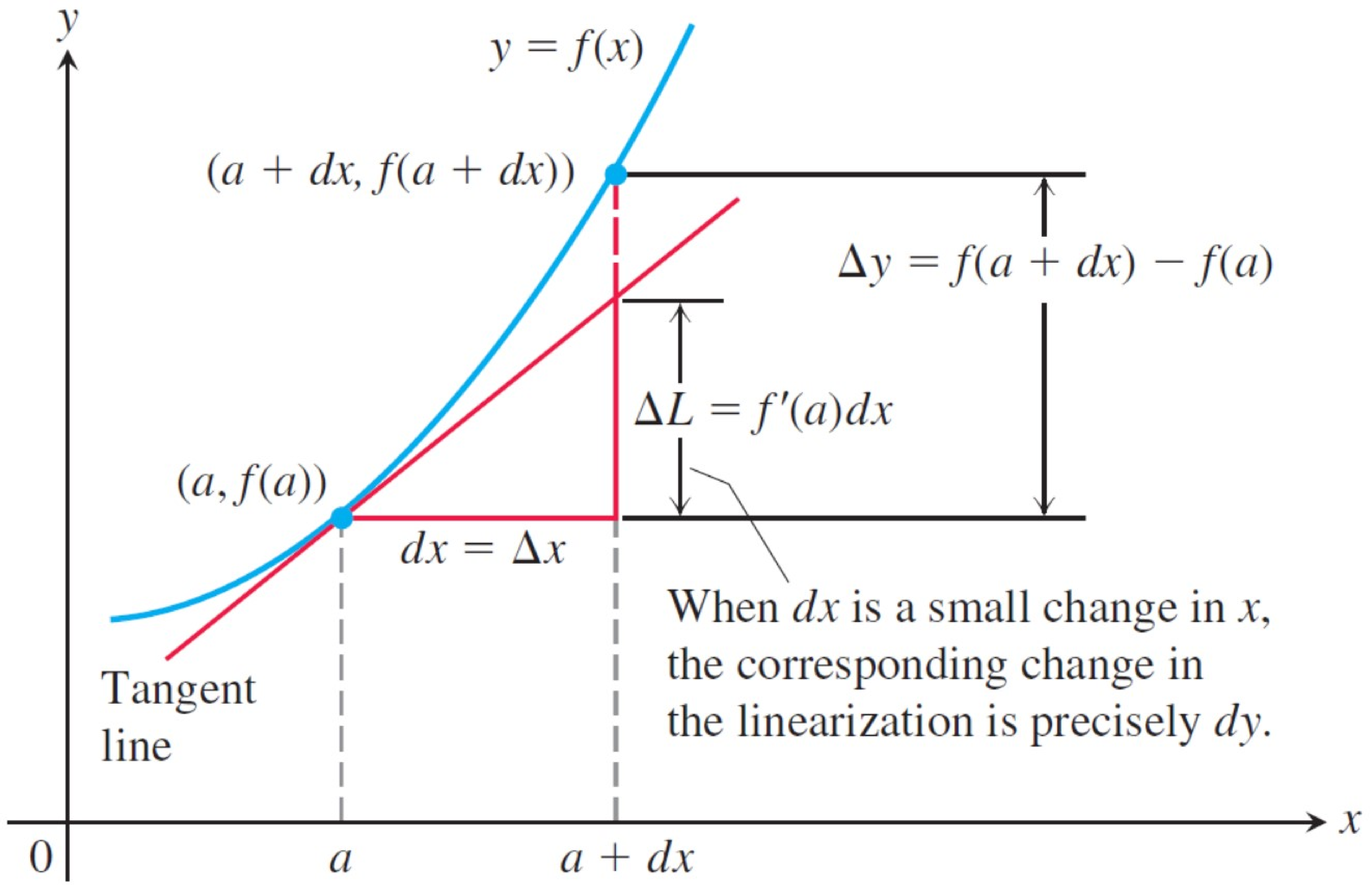
(linear approximation)

a = center of approximation.

$L(x)$ = a linear function

satisfying "$f(x) \cong L(x)$ near a"

(the)



y
$y = f(x)$
$(a + dx, f(a + dx))$
$\Delta y = f(a + dx) - f(a)$
$\Delta L = f'(a)dx$
$(a, f(a))$
$dx = \Delta x$
When dx is a small change in x, the corresponding change in the linearization is precisely dy.
Tangent line
x
0
a
$a + dx$

Eg 1: Find linearization of $(1+x)^k$ near $x=0$, $(k \in \mathbb{R})$

<u>Ans</u>: $f(0) = 1$

$f'(0) = k \cdot (1+0)^{k-1} = k$

$\therefore L(x) = 1 + kx$.

Eg 2: find an approximate value of $\sqrt{1.001}$

<u>Ans</u>: $(1+x)^k$, $k=\frac{1}{2}$, $x=0.01$

$\therefore \sqrt{1.001} \cong 1 + \frac{1}{2} 0.001$

$= 1.0005$

Eg3 $(7.97)^{\frac{1}{3}} \cong ?$

$$\underline{\text{Ans}} = (8-0.03)^{\frac{1}{3}}$$

$$= \left(8\left(1-\frac{0.03}{8}\right)\right)^{\frac{1}{3}}$$

$$\cong 2\left(1-\frac{1}{3}\cdot\frac{0.03}{8}\right)$$

$$= 1.9975$$

Eg4. $\sin\left(\frac{\pi}{6}+0.001\right) \cong ?$

$\underline{\text{Ans}}$: Let $f(x) = \sin(x)$

$$L(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\cdot\left(x-\frac{\pi}{6}\right)$$

$$\text{Ans} = L\left(\frac{\pi}{6}+0.001\right) = \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\cdot 0.001$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\cdot 0.001$$

What does it mean by "$f(x) \doteqdot L(x)$ near a"?

$$f(x) = L(x) + \text{error}$$

<u>Prop</u>: If f is diff. at $x=a$

then $$\lim_{x-a} \frac{f(x)-L(x)}{x-a} = 0$$

<u>Pf</u>: $$= \lim_{x\to a} \frac{f(x)-f(a)-f'(a)(x-a)}{x-a}$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} - \lim_{x\to a} \frac{f'(a)(x-a)}{(x-a)}$$

$$= f'(a) - f'(a) = 0$$

In summary, if f is diff. at $x=a$, then

$$\lim_{x\to a} \frac{\overbrace{\left(f(x)-L(x)\right)}^{\dot{=}E(x)}}{x-a} = 0$$

i.e. $$\lim_{x\to a} \frac{E(x)}{x-a} = 0$$

$$\therefore\ \underline{E(x)} = \frac{E(x)}{x-a}\cdot(x-a)$$

$$= \underbrace{\left(\begin{array}{c}\text{Something that}\\ \text{goes to zero,}\\ \text{as } x\to a\end{array}\right)}_{\equiv\,\varepsilon}(x-a)$$

EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$ (Figure 3.52).

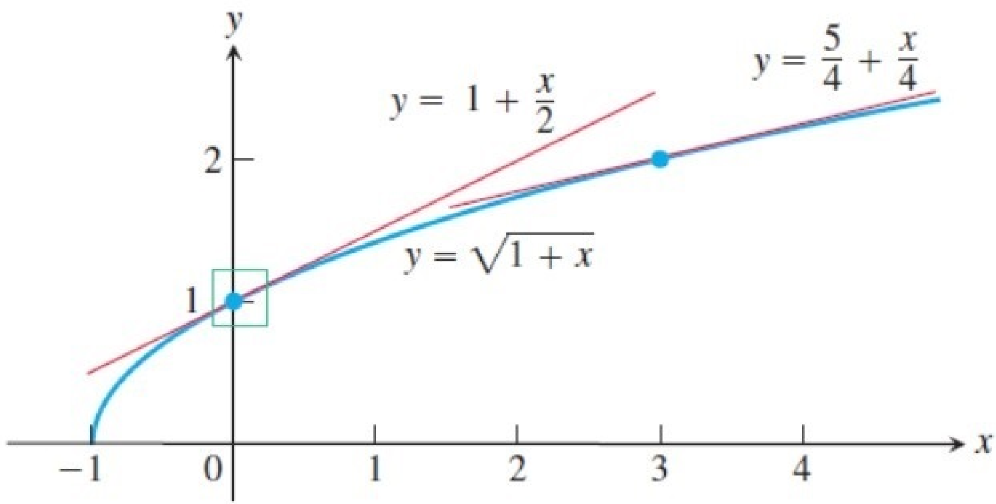


FIGURE 3.52 The graph of $y = \sqrt{1+x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.53 shows a magnified view of the small window about 1 on the y-axis.

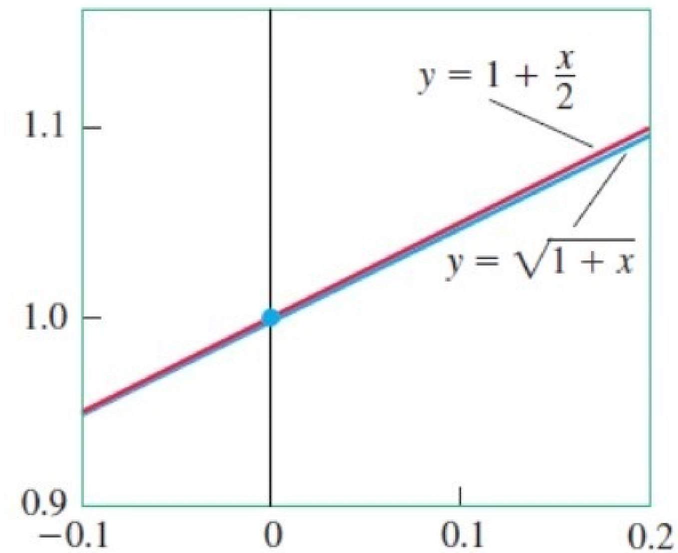


FIGURE 3.53 Magnified view of the window in Figure 3.52.

	L(x)	f(x)	$\lvert f(x)-L(x) \rvert = \lvert \Delta y - \Delta L \rvert = \lvert E(x) \rvert$	$\lvert \Delta y - \Delta L \rvert / \Delta x = \epsilon$
Δx	**Approximation**	**True value**	**|True value − approximation|**	
0.2	$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$	0.022775
0.05	$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$	0.0061
0.005	$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$	0.0006

$f(x)$ is diff. at $x=a$

(This '<=' is trivial since L(x) contains f'(a))

$$\Leftrightarrow f(x) = L(x) + \varepsilon \cdot (x-a), \quad \lim_{x \to a} \varepsilon = 0$$

$$\Leftrightarrow \Delta y = f'(a)\Delta x + \varepsilon \cdot (x-a) \quad \lim_{x \to a} \varepsilon = 0$$

$$(\Delta x = x - a, \ \Delta y = f(x) - f(a))$$

$$\Leftrightarrow \frac{\Delta y}{\Delta x} = f'(a) + \varepsilon, \quad \lim_{x \to a} \varepsilon = 0$$

pf of Chain Rule:

$$\lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} =$$

$$= \lim_{u \to g(x_0)} \frac{f(u) - f(g(x_0))}{u - g(x_0)} \cdot \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

Let $\Delta x = x - x_0$, $\Delta u = g(x) - g(x_0)$

$$= u - u_0$$

$$\Delta y = f(g(x)) - f(g(x_0)) = f(u) - f(u_0)$$

Old proof: $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$

let $\Delta x \to 0$

problem: devide by zero if $g(x) = g(x_0)$

(while $x \neq x_0$)

To fix the problem

we change $\lim\limits_{\Delta u \to 0} \dfrac{\Delta y}{\Delta u} = f'(u_0)$

to $\begin{cases} \Delta y = f'(u_0)\Delta u + \varepsilon_2 \Delta u \\ \lim\limits_{\Delta u \to 0} \varepsilon_2 = 0 \end{cases}$

Similarly, $\begin{cases} \Delta u = g'(x_0)\Delta x + \varepsilon_1 \Delta x \\ \lim\limits_{\Delta x \to 0} \varepsilon_1 = 0 \end{cases}$

$$\therefore \Delta y = (f'(u_0) + \varepsilon_2)\Delta u$$
$$= (f'(u_0) + \varepsilon_2)(g'(x_0) + \varepsilon_1)\Delta x$$

Note: $\Delta x \to 0 \Longrightarrow \Delta u \to 0 \Rightarrow \lim\limits_{\Delta x \to 0} \varepsilon_2 = 0$

Let $\Delta x \to 0$

$$\Rightarrow \Delta y = f'(u_0) \cdot g'(x_0) \cdot \Delta x$$

$$+ (\underbrace{f'(u_0)\varepsilon_1 + g'(x_0)\varepsilon_2 + \varepsilon_1 \varepsilon_2}_{= \varepsilon}) \Delta x$$

$$= f'(u_0) g'(x_0) \Delta x + \varepsilon \Delta x$$

$$\lim_{\Delta x \to 0} \varepsilon = 0$$

$$\therefore \frac{dy}{dx} = f'(u_0) g'(x_0)$$

i.e. $\frac{d}{dx} f(g(x)) = f'(g(x_0)) \cdot g'(x_0)$

Rm

$g(x)$ is 1st order approximation of $f(x)$ near $x=a$ (and vice versa)

(i.e. $y=g(x)$ is tangent to $y=f(x)$ at $x=a$)

if (i) $g(a)=f(a)$

(ii) $\displaystyle\lim_{x\to a}\frac{f(x)-g(x)}{x-a}=0$

If $g(x)=L(x)=C+m(x-a)$

(HW66 $\Longrightarrow$ $\begin{cases} C=f(a) \\ m=f'(a) \end{cases}$)

and $L(x)$ is linearization of $f(x)$ near $x=a$
(linear approximation)

<u>Def</u>: If $f(x)$ is twice differentiable and $Q(x)$ is a quadratic approximation of $f(x)$ near $x=a$ (and vice versa)

if (i) $f(a) = Q(a)$

(ii) $\displaystyle\lim_{x \to 0} \frac{f(x) - Q(x)}{(x-a)^2} = 0$

<u>HW55</u>: If $Q(x)$ is a quadratic polynomial, then

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$